A STOKESIAN SUBMARINE

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Abstract. We consider the problem of swimming at low Reynolds numbers. This is the relevant asymptotic for micro- and nano-robots needing to navigate in an aqueous medium. As a model, we propose a robot composed of three balls. The relative positions of these balls can change according to three degrees of freedom. We prove that this robot is able to navigate in a plane by modifying the conformation of its shape.

Résumé. Nous considérons le problème de la nage à faible nombre de Reynolds. C'est l'asymptotique pertinente pour les micro- et nano-robots devant se déplacer dans un milieu aqueux. Nous proposons comme modèle un robot formé de trois boules qui peuvent se déplacer les unes par rapport aux autres selon trois degrés de liberté. Nous démontrons qu'en changeant sa conformation, ce robot peut effectivement naviguer dans un plan.

1. Introduction

In this note we are concerned with the problem of self-propulsion at low Reynolds numbers. Namely, is it possible for a swimmer submerged in a Stokes fluid to move solely by changing the configuration of its shape ? Recent studies have shown that a solid can swim by prescribing the velocity of the fluid on its surface, see [10]. Here we assume that the velocities of the fluid and of the swimmer are equal on its boundary.

Swimming at low Reynolds numbers is the problem faced by microorganisms such as bacteria, unicellular eukaryotes or special cells of multicellular organisms such as sperm cells. A lot of papers are dedicated to this subject in the biological literature. We refer to the recent paper [6] for a description of known biological swimming mechanisms and an overview of the subject.

From the physicist's point of view, the subject has been initiated by Taylor [14], Lighthill [7] and later Purcell [9] and Shapere and Wilczek [12,13]. The motion of a micro-swimmer is dominated by viscosity and inertia is negligible. In this situation, it turns out that the instantaneous global displacement of the swimmer is a linear function of its deformation. In particular, the total displacement of the swimmer only depends on the assumed sequence of shapes and not on the velocity this sequence has been ran through. Moreover a reciprocal deformation does not lead to any displacement and then at low Reynolds numbers, an efficient swimmer has to perform non-reciprocal strokes. Consequently, at small scales, swimming is not an obvious task. However some model swimmers, being able to move along a line, have been proposed [4,8,9].

The swimming issue is not only theoretical. Low Reynolds numbers are also the right asymptotic regime for swimming micro- or nano-robots. In the near future we can imagine building some micro-devices that navigate and interact with biological systems. Very recently, a 250 μm-length micro-motor has been designed [15]

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opening the possibility for practical micro-surgeons. In this direction it is interesting to study simple devices that are able not only to translate but also to navigate. By navigating we mean translating and rotating.

We will introduce below such a model robot composed of three coplanar spheres and prove its ability to navigate in a plane. The proof is similar to the one proposed in [2] in the sense that the question of whether the device is able to navigate reduces to a controllability problem. As in [2], local controllability is obtained for large values of the control parameters. However, here we deal with 3 control parameters and 2D displacements rather than 2 control parameters and 1D displacements. Moreover, the asymptotic analysis has been drastically simplified and now allows to consider devices made of an arbitrary number of spheres.

In the rest of this introduction, we set the swimming problem, briefly describe existing micro-swimmers and finally present our model and main result.

Swimming at low Reynolds numbers

At small scales and small velocities, inertia is negligible compared to viscosity. If K is the region filled by a swimmer and $\Gamma = \partial K$ its boundary, we can assume that the velocity and pressure v, p in the surrounding fluid solve the Stokes equations:

$$\begin{cases} -\eta \Delta u + \nabla p & = & 0 & \text{in } \mathbf{R}^3 \setminus K, \\ \nabla \cdot u & = & 0 & \text{in } \mathbf{R}^3 \setminus K, \\ u & = & g & \text{on } \Gamma, \\ u, p & \to & 0 & \text{at } \infty \end{cases} \tag{1.1}$$

The boundary velocity g is the sum of a contribution due to the deformation of the swimmer and of a contribution due to the rigid induced displacement. Let us describe these contributions and introduce some notations.

As noticed in [11], since we neglect inertia, there is no canonical way to attach a center and a set of axes to the swimmer. We have to choose arbitrarily a standard position and an orientation for each shape. In the sequel we only consider families of shapes characterized by a finite number of parameters $l = (l_1, \cdots, l_n)$ varying in an open set $\mathcal{L} \subset \mathbf{R}^n$. The standard positions of the shapes we consider are given by a family (γ_l) of homeomorphisms:

$$\gamma_l \; : \; \Gamma \; \longrightarrow \Gamma_l,$$

where Γ is a reference shape. The position x of a point of the boundary is then given by

$$x \; = \; c + \theta \gamma_l(x_0),$$

where x_0 is a point of the fixed reference shape Γ and $(c, \theta) \in \mathbf{R}^3 \times \mathrm{SO}_3(\mathbf{R})$ characterize the solid displacement. Next, the velocity of the boundary at this point writes

$$g(x) \; = \; v + \omega \times x + \theta w(x), \tag{1.2}$$

where $v, \omega \in \mathbf{R}^3$ are the instantaneous speed and rotation of the solid referential and where $w \; = \; (\dot{l} \cdot \nabla) \gamma_l(x_0)$ corresponds to the velocity due to the deformation of the swimmer.

The surface density of the forces applied by the swimmer on the fluid on Γ_l is given by the formula $f(x) \; = \; \sigma(x) \boldsymbol{n}(x)$, where $\sigma := \eta(\nabla u + \nabla u^t) - pId$ is the Cauchy stress tensor in the fluid, $\boldsymbol{n}$ is the inner unit normal to K on Γ_l and (u, p) solve (1.1) with boundary data (1.2). If no external force is applied on the fluid-swimmer system, neglecting inertia, the total force and torque exerted by the swimmer on the fluid must vanish. We obtain the following system of six conditions

$$\int_{\Gamma_l} \sigma \boldsymbol{n}(x) \, dx \; = \; 0, \qquad \int_{\Gamma_l} x \times \sigma \boldsymbol{n}(x) \, dx \; = \; 0. \tag{1.3}$$

Solving these equations we obtain the instantaneous solid displacement as a linear function of $\dot{l}$:

$$v \; = \; V_{l,c,\theta} \dot{l}, \qquad \omega \; = \; \Omega_{l,c,\theta} \dot{l}. \tag{1.4}$$

Notice that the coefficients depend non-linearly on the shape parameter l.

We will call stroke any closed path $\{l(t), t \in [0, T]\}$ in the set of parameters and say that our robot is able to swim if there exists a stroke leading to a non-vanishing net displacement $c(T) - c(0) = \int_0^T v$.

A well known result called the Scallop Theorem (see e.g. [2,6,9,13,14]) states that a stroke in a one parameter family of shapes does not lead to any displacement. Indeed, in such a case, we may write

$$\int_0^T v = \int_0^T G_{l_1(s)} \dot{l}_1(s)\, ds = \oint G_{l_1}\, dl_1 = 0.$$

Examples of swimmers

Swimming at low Reynolds number is thus very far from the intuition we usually develop at our scales. Anyway, there exist some model robots which are known to be able to swim. We briefly describe three of them. Each one corresponds to a two-parameter family of shapes.
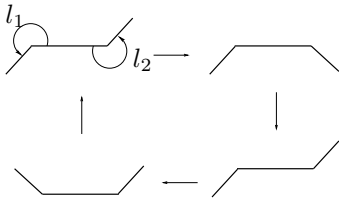

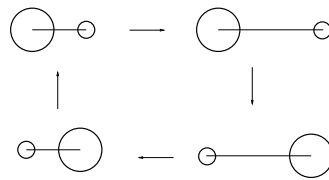FIGURE 1. a stroke for the three-link swimmer of Purcell

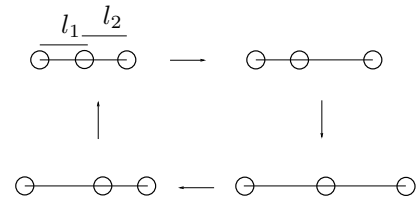FIGURE 2. a stroke for the pushmepullyou swimmer

FIGURE 3. a stroke for the three-sphere swimmer

The first one was proposed by Purcell [9]. It is an object composed of three solid lines linked by two hinges. The two shape parameters are the angles l_1 and l_2 between two consecutive lines. By varying alternatively these angles we obtain the non-reciprocal stroke shown Fig. 1. In that case, the scallop Theorem does not apply and this stroke can lead to a net displacement along the horizontal axis.

As a second example, we present the pushmepullyou swimmer of Avron, Kenneth and Oakim [4]. It is made of the union of two spheres S_1, S_2. The two parameters are the distance between the centers of the spheres and the ratio between their volume, the total volume being fixed — see Fig. 2.

Our last example is the three-sphere swimmer of Najafi and Golestanian [8]. The robot is composed of three identical aligned spheres. The shape parameters are the distances between two consecutive spheres — see Fig. 3.

These three robots are known to be able to swim — see [2, 4, 8].

In these examples, we obtain a displacement along one direction. Let us now introduce a new simple robot being able to navigate (translate and rotate) in a plane.

A three-ball submarine

Our robot is composed of three balls B_1, B_2, B_3 of radii $a > 0$. The structure of the robot is built on three rays $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3 \subset \{z = 0\}$ spanned by the three vertices of an equilateral triangle centered at 0 — see Fig. 4. For $i = 1, 2, 3$, the center of the ball B_i can translate freely along the ray $\mathcal{D}_i$ but we assume that the balls do not rotate in the frame attached to $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3$. As a consequence, the shape of the robot is characterized by the three distances l_1, l_2, l_3 between the centers of the balls and the origin.

Of course, by symmetry, this robot can not move in the z direction. For the same reason, it can not rotate around a non-vertical axis. The state of the three-ball submarine is determined by the three lengths l_1, l_2, l_3, the coordinates c_1, c_2 of the intersection of $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3$ and the algebraic angle θ between e_x and $\mathcal{D}_1$. In this particular case, we may rewrite (1.4) as

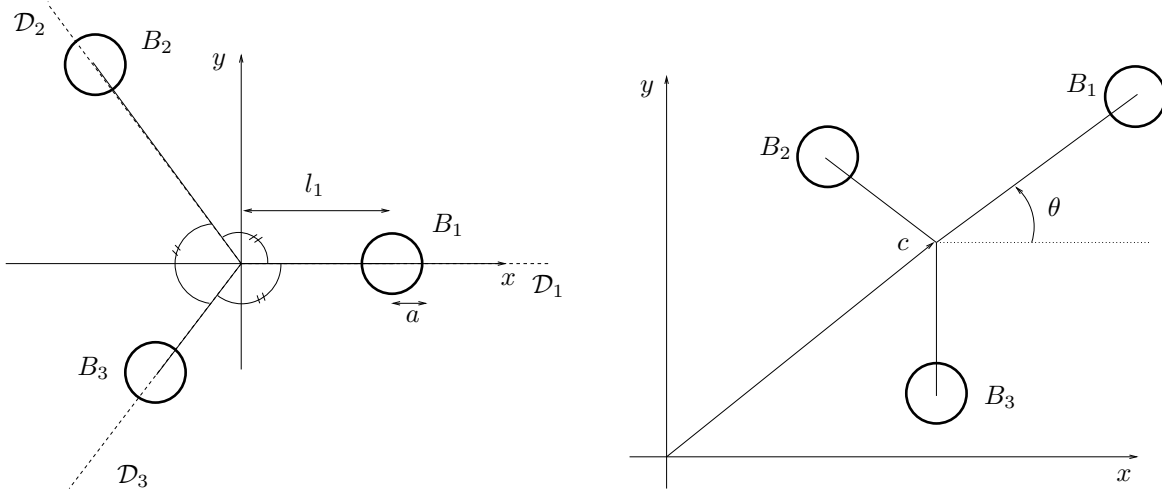$$\dot{c} = V_{l,c,\theta}\dot{l}, \qquad \dot{\theta} = \Omega_{l,c,\theta}\dot{l}.$$

FIGURE 4. The three-ball submarine

Writing $X = (l, c, \theta)$, the evolution of the state of the submarine is given by the relation

$$\dot{X} \;=\; F_X \dot{l} \;:=\; (\dot{l}, V_X \dot{l}, \Omega_X \dot{l}). \tag{1.5}$$

Our main result states that if the parameters are allowed to vary in the neighborhood of $\{l_1 = l_2 = l_3 = 1/\varepsilon\}$ with $\varepsilon > 0$ small enough, then the three-ball submarine can achieve any small displacement in the plane.

Theorem 1.1. *For $\varepsilon > 0$ small enough, there exists a neighborhood $\mathcal{N}$ of $X_\varepsilon := ((1/\varepsilon, 1/\varepsilon, 1/\varepsilon), 0, 0, 0)$ in $(0, +\infty)^3 \times \mathbf{R}^2 \times \mathbf{R}$ such that for any $X_i, X_f \in \mathcal{N}$, there exists a solution to (1.5) with end points X_i, X_f.*

By translation and rotation invariance of the problem, we see that repeating small strokes, our submarine can achieve any translation and any rotation in the plane.

A controllability problem

Following the method of [2], we rephrase Theorem 1.1 in the language of control theory. In this setting, Theorem 1.1 claims that system (1.5) is locally controllable at X_ε for ε small enough.

Let F_X^1, F_X^2, F_X^3 denote the rows of the matrix F_X, i.e: $F_X^i := F_X e_i$ where (e_1, e_2, e_3) is the canonical basis of $\mathbf{R}^3$. Let $[F_X^i, F_X^j]$ denotes the Lie bracket $(F_X^i \cdot \nabla) F_X^j - (F_X^j \cdot \nabla) F_X^i$. Chow's Theorem gives a sufficient condition for local controllability (see e.g. [1]) is:

$$\mathrm{span} \left\langle F_{X_\varepsilon}^1, F_{X_\varepsilon}^2, F_{X_\varepsilon}^3, \ [F_X^i, F_X^j]_{|X=X_\varepsilon}, 1 \le i < j \le 3 \right\rangle \;=\; \mathbf{R}^3 \times \mathbf{R}^2 \times \mathbf{R}. \tag{1.6}$$

Writing $F_X^k = (e_k, T_X^k)$, with $T_X^k := (V_X^k, \Omega_X^k) \in \mathbf{R}^3$, we easily compute that this condition is equivalent to

$$\det \left((F_X^1 \cdot \nabla) T_X^2 - (F_X^2 \cdot \nabla) T_X^1, \ (F_X^2 \cdot \nabla) T_X^3 - (F_X^3 \cdot \nabla) T_X^2, \ (F_X^3 \cdot \nabla) T_X^1 - (F_X^1 \cdot \nabla) T_X^3 \right)_{|X=X_\varepsilon} \;\ne\; 0. \tag{1.7}$$

Remark 1.2. If $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3$ are not assumed to be coplanar, we expect that any solid displacement in $\mathbf{R}^3$ can be achieved (and not only planar displacements). In that case we should add iterated Lie brackets of the form $[F_X^i, [F_X^j, F_X^k]]$ in condition (1.6).

Remark 1.3. Once we know that the robot is able to navigate, we may ask as in e.g. [2, 3] which are the optimal strokes for a fixed global displacement in term of energy expense. This question will be addressed in a forthcoming paper.

In Section 2 we briefly recall well known results on the Stokes problem (1.1) in the complement of n balls. We also establish an asymptotic result for this Stokes problem in the regime of large distances between balls. In Section 3 we prove Theorem 1.1

2. STOKES PROBLEM IN THE COMPLEMENT OF n BALLS

In this section, we consider the Stokes problem in the complement of n balls of radii $a > 0$.

In the sequel B is the closed ball of radius a centered at 0 and $B_y := y + B$ denotes the ball centered at a point $y \in \mathbf{R}^3$. A configuration of n balls is characterized by their centers $x = (x_1, \cdots, x_n) \in (\mathbf{R}^3)^n$. We only consider configurations of non-overlapping balls by imposing $x \in \mathcal{C}_n$ where

$$\mathcal{C}_n := \left\{ y \in (\mathbf{R}^3)^n \ : \ R(y) > 2a \right\}, \qquad \text{with} \qquad R(y) := \min_{i \neq j} |y_i - y_j|.$$

We now fix $x \in \mathcal{C}_n$. The domain filled by the fluid is $\Omega_x := \mathbf{R}^3 \setminus (\cup_{1 \leq i \leq n} B_{x_i})$. Let us introduce the Hilbert space $\mathcal{H}^{1/2} := H^{1/2}(\partial B, \mathbf{R}^3)$ and its dual $\mathcal{H}^{-1/2} := H^{-1/2}(\partial B, \mathbf{R}^3)$. We also define the Cartesian products

$$\mathcal{H}_n^{1/2} := \{g = (g_1, \cdots, g_n) \ : \ g_i \in \mathcal{H}^{1/2}\}, \qquad \mathcal{H}_n^{-1/2} := \{f = (f_1, \cdots, f_n) \ : \ f_i \in \mathcal{H}^{-1/2}\}.$$

Let $g \in \mathcal{H}_n^{1/2}$, the Stokes problem in Ω_x, with Dirichlet data g_i on ∂B_i reads

$$\begin{cases} -\eta \Delta u + \nabla p &= \ 0, & \text{in } \Omega_x, \\ \nabla \cdot u &= \ 0 & \text{in } \Omega_x, \\ u &= \ g_i(\cdot - x_i) & \text{on } \partial B_{x_i}, \text{ for } i = 1, \cdots, n, \\ u, p &\to \ 0 & \text{at } \infty \end{cases} \tag{2.1}$$

It is well known (see e.g. [5] p. 154) that Problem (2.1) admits a unique solution $(u, p) \in \mathcal{V}_x \times L^2(\Omega_x)$ where

$$\mathcal{V}_x := \left\{ v \in \mathcal{D}'(\Omega_x, \mathbf{R}^3) \ : \ \nabla v, \frac{v}{\sqrt{1 + |r|^2}} \in L^2(\Omega_x) \right\}.$$

Moreover we have $\sigma \cdot \boldsymbol{n} \in H^{-1/2}(\partial \Omega_x, \mathbf{R}^3)$ where $\sigma := \eta(\nabla u + \nabla u^t) - pId$ is the Cauchy stress tensor, and $\boldsymbol{n}$ is the outer unit normal to $\partial \Omega_x$. From this, for $i = 1, \cdots, n$ we can define $f_i \in \mathcal{H}^{-1/2}$ by $f_i := \{\sigma \cdot n\} (x_i + \cdot)$. We have thus defined a bounded linear application

$$\Phi_x \ : \ g \in \mathcal{H}_n^{1/2} \ \longmapsto \ f \in \mathcal{H}_n^{-1/2} \tag{2.2}$$

associating to a velocity data on the boundary of the balls the resulting surface force density applied by the balls on the fluid. This application is usually called the Dirichlet to Neumann map (for the Stokes problem).

It turns out that the inverse problem is easier since we have an explicit formula for the velocity field u as a convolution of f with the fundamental solution of Stokes equation:

$$u(r) \ = \ \sum_{i=1}^{n} \int_{\partial B_i} f_i(r' - x_i) G(r - r') \, dr', \qquad \text{with} \quad G(r) := \frac{1}{8\pi\eta} \left(\frac{1}{|r|} + \frac{r \otimes r}{|r|^3} \right). \tag{2.3}$$

The integrals in the formula have a meaning for $r \in \Omega_x$ and $f \in \mathcal{H}_n^{-1/2}$ through the duality product $\mathcal{H}^{1/2} - \mathcal{H}^{-1/2}$. On the boundary of the ball B_i, this formula leads to

$$g_i(r) \ = \ \Phi_0^{-1} f_i(r) + \sum_{j \neq i} \int_{\partial B} f_i(r') G(r - r' + x_j - x_i) \, dr', \qquad i = 1, \cdots, n, \quad \forall r \in \partial B, \tag{2.4}$$

where Φ_0 is the application defined by (2.2) in the case $n = 1$ — notice that by translation invariance this application does not depend on the position of the ball.

Large distances approximation

For later use we are interested in the asymptotic behavior of Φ_x and $\nabla_x \Phi_x$ as $R(x) := \min_{i<j} |x_i - x_j|$ tends to infinity. Let us introduce the notations

$$x_{i,j} := x_i - x_j, \qquad r_{i,j} := |x_{i,j}| \qquad \text{and} \qquad e_{i,j} := \frac{x_{i,j}}{r_{i,j}}.$$

We first prove

Proposition 2.1. *For every $f \in \mathcal{H}_n^{-1/2}$, we have*

$$(\Phi_x^{-1} f)_i = \Phi_0^{-1} f_i + \frac{1}{8\pi\eta} \sum_{j \neq i} \frac{1}{r_{i,j}} (Id + e_{i,j} \otimes e_{i,j}) \int_{\partial B} f_j + \psi_{x,i} f,$$

with $\qquad \|\psi_x\|_{\mathcal{L}(\mathcal{H}_n^{-1/2}, \mathcal{H}_n^{1/2})} \lesssim R^{-2}(x), \qquad$ *and* $\qquad \|\nabla_x \psi_x\|_{\mathcal{L}(\mathcal{H}_n^{-1/2}, \mathcal{H}_n^{1/2})^{3n}} \lesssim R^{-3}(x).$

Proof. Let $x \in \mathcal{C}_n$ and $f \in \mathcal{H}_n^{-1/2}$. From (2.2) (2.4) we have for $i = 1, \cdots, n$ and $r \in \partial B$,

$$(\Phi_x^{-1} f)_i(r) = \Phi_0^{-1} f_i(r) + \sum_{j \neq i} \int_{\partial B} f_i(r') G(r - r' + x_{ij}) \, dr'. \tag{2.5}$$

Using the expansion $\quad G(x_{i,j} + z) = G(x_{i,j}) + \{G(x_{i,j} + z) - G(x_{i,j})\}$, for $|z| \leq 2a$, we get

$$G(x_{i,j} + z) = G(x_{i,j}) + \tilde{G}(x_{i,j}, z) \tag{2.6}$$

where the functions $r_{i,j}^2 \tilde{G}(x_{i,j}, \cdot)$ and $r_{i,j}^3 \nabla_x \tilde{G}(x_{i,j}, \cdot)$ remain bounded in any $C^k(2B)$ as $r_{i,j}$ tends to infinity. Plugging (2.6) in (2.5) yields the Proposition. $\qquad \square$

Since the operator $f \in \mathcal{H}_n^{-1/2} \mapsto (\Phi_0^{-1} f_1, \cdots, \Phi_0^{-1} f_n) \in \mathcal{H}_n^{1/2}$ is an isomorphism, we deduce the following asymptotic expansion for Φ_x.

Proposition 2.2. *For every $g \in \mathcal{H}_n^{1/2}$,*

$$(\Phi_x g)_i = \Phi_0 \left\{ g_i - \frac{1}{8\pi\eta} \sum_{j \neq i} \frac{1}{r_{i,j}} (Id + e_{i,j} \otimes e_{i,j}) \int_{\partial B} \Phi_0 g_j \right\} + \phi_{x,i} g,$$

with $\qquad \|\phi_x\|_{\mathcal{L}(\mathcal{H}_n^{1/2}, \mathcal{H}_n^{-1/2})} \lesssim R^{-2}(x), \qquad$ *and* $\qquad \|\nabla_x \phi_x\|_{\mathcal{L}(\mathcal{H}_n^{1/2}, \mathcal{H}_n^{-1/2})^{3n}} \lesssim R^{-3}(x).$

Rigid motion of n independent balls in a Stokes fluid

We consider Problem (2.1) with a boundary data $g = (g_1, \cdots, g_n)$ generated by independent rigid motions of the n balls, i.e, there exists $v = (v_1, \cdots, v_n)$ and $\omega = (\omega_1, \cdots, \omega_n)$ in $(\mathbf{R}^3)^n$ such that

$$g_i(r) = v_i + \omega_i \times r, \qquad \text{for } i = 1, \cdots, n, \quad r \in \partial B. \tag{2.7}$$

We are interested in the force field $f = (f_1, \cdots, f_n)$ generated by these velocities on the boundaries of $B_{x_1}, \cdots, B_{x_n}$, i.e $f = \Phi_x g$.

In the case $n = 1$, the solution is well known: we have

$$\Phi_0 g_1(r) \;=\; \frac{\eta}{a}\left(\frac{3}{2}v_1 + 3\omega_1 \times r\right).$$

Using this formula and Proposition 2.2, we end with

Proposition 2.3. *Let $x \in \mathcal{C}_n$ and g of the form (2.7), we have*

$$(\Phi_x g)_i \;=\; \frac{3\eta}{2a}\left(v_i + 2\omega_i \times r - \frac{3}{4}\sum_{j\neq i}\frac{a}{r_{i,j}}(v_j + (e_{i,j}\cdot v_j)e_{i,j})\right) + \varphi_{x,i}(v,\omega),$$

with $\qquad \|\varphi_x(v,\omega)\|_{\mathcal{H}_n^{-1/2}} \;\lesssim\; R^{-2}(x)|v,\omega|, \qquad$ *and* $\qquad \|\nabla_x\varphi_x(v,\omega)\|_{\mathcal{H}_n^{-1/2}} \;\lesssim\; R^{-3}(x)|v,\omega|.$

Notice that since $\int_{\partial B}\omega \times r\,dr = 0$, rotations do not appear in the $O(R^{-1}(x))$ terms of the asymptotic.

3. Local controllability of the three-ball submarine

In this section we prove Theorem 1.1: the three ball submarine is locally controllable in the neighborhood of $l = (1/\varepsilon, 1/\varepsilon, 1/\varepsilon)$. We have seen that it were sufficient to establish (1.7). To do so, in the following, we write the self propulsion conditions (1.3) for $X_\varepsilon = ((1/\varepsilon, 1/\varepsilon, 1/\varepsilon), 0, 0, 0)$. Then using Proposition 2.3 we compute the leading order of $F_{X_\varepsilon}^i$ as ε tends to 0.

Next, using again Proposition 2.3 and conditions (1.3), we find the leading order term of $(F_X^i \cdot \nabla)T_X^j$ at $X = X_\varepsilon$.

Finally, we compute the leading order of the determinant in equation (1.7) and conclude.

Conventions and Notations

First we introduce some notations in order to describe the motion of the three ball submarine. For $i = 1, 2, 3$, let x_i be the center of the ball B_i and p_i the unit vector

$$p_i \;:=\; \frac{x_i - c}{|x_i - c|}.$$

By symmetry the points x_i and c remain in the plane $\{z = 0\}$. In the sequel we identify $\mathbf{R}^2$ with the plane $\{z = 0\} \subset \mathbf{R}^3$ and we consider that x_i, c and p_i are 2-dimensional vectors. For $x = (x^1, x^2) \in \mathbf{R}^2$, we will write

$$x^\perp \;:=\; (-x^2, x^1)$$

to denote the $\pi/2$-rotation of x.

The position x_i may be written

$$x_i \;=\; c + l_i p_i$$

The velocity of ball B_i results from the rigid motion of the frame attached to $(c, p_1, p_1^\perp) \subset \{z = 0\}$ and from the translation of the ball in this frame. The instantaneous motion of the solid frame is characterized by the velocity v of the center c and by an angular velocity $\omega \in \mathbf{R}$. With these notations the velocity of x_i writes

$$\dot{x}_i \;=\; v + \dot{l}_i p_i + \omega l_i p_i^\perp, \qquad\qquad i = 1, 2, 3,$$

Since the motion of B_{x_i} in the frame $(c, p_1, p_1^{\perp})$ is a pure translation in the direction p_i, the fluid velocity on ∂B_{x_i} is constrained to be equal to

$$g_i(r) \;=\; \underbrace{v + \omega(l_i p_i + r)^{\perp}}_{=:\; h_i(v,\omega)(r)} \;+\; \underbrace{\dot{l}_i p_i}_{=:\; k_i(r)}, \qquad r \in \partial B, \quad i = 1, 2, 3, \tag{3.1}$$

where k stands for the velocity of the point attached to the solid frame and h for the relative velocity of the balls in this frame.

In order to achieve the computations below we introduce the unit vectors $e_{i,j} \;:=\; \dfrac{x_i - x_j}{|x_i - x_j|}$ and the distances $r_{i,j} := |x_i - x_j|$.
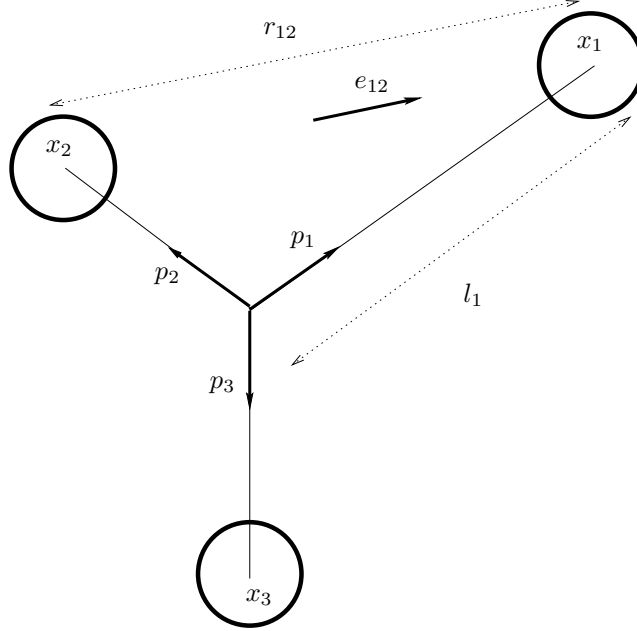

FIGURE 5. Notations for the three-ball submarine

Finally, since we are interested in the regime of large distances between the balls, we perform the change of variables

$$\lambda \;:=\; \varepsilon l, \qquad \rho_{i,j} \;:=\; \varepsilon r_{i,j}, \qquad \tilde{v} \;:=\; \varepsilon v. \tag{3.2}$$

Self propulsion

We now write the self propulsion conditions (1.3). By symmetry the vertical component of the total force exerted on the fluid and the horizontal components of the torque vanish so we only keep three conditions:

$$e_i \cdot \sum_{i=1}^{3} \int_{\partial B_i} \sigma \boldsymbol{n} \;=\; 0, \qquad \text{for } i = 1, 2 \qquad \text{and} \qquad e_3 \cdot \sum_{i=1}^{3} \int_{\partial B_i} r \times \sigma \boldsymbol{n} \; dr \;=\; 0. \tag{3.3}$$

The forces exerted on the fluid on ∂B_i are given by $\sigma \boldsymbol{n} \; = \; (\Phi_x g)_i = (\Phi_x h)_i + (\Phi_x k)_i$. Using the asymptotic given in proposition 2.3, we have

$$(\Phi_x h)_i \;=\; \frac{3\eta}{2a}\left(v + \omega l_i p_i^\perp + 2\omega r^\perp - \frac{3}{4}\sum_{j,j\neq i}\frac{a}{r_{i,j}}\left\{(v + \omega l_j p_j^\perp) + (e_{i,j}\cdot(v + \omega l_j p_j^\perp))e_{i,j}\right\}\right)$$
$$+ s_{x,i}((v + \omega l_j p_j^\perp)_j, \omega),$$

$$(\Phi_x k)_i \;=\; \frac{3\eta}{2a}\left(\dot{l}_i p_i - \frac{3}{4}\sum_{j,j\neq i}\frac{a}{r_{i,j}}\dot{l}_j\,(p_j + (e_{i,j}\cdot p_j)e_{i,j})\right) + t_{x,i}((\dot{l}_j p_j)_j),$$

where $s_{x,1}, \cdots, s_{x,3}, t_{x,1}, \cdots, t_{x,3}$ are linear operators with values in $\mathcal{H}_n^{-1/2}$ satisfying

$$\|s_x\| \;\lesssim\; R^{-2}(x), \qquad \|\nabla_x s_x\| \;\lesssim\; R^{-3}(x) \qquad \text{and} \qquad \|t_x\| \;\lesssim\; R^{-2}(x), \qquad \|\nabla_x t_x\| \;\lesssim\; R^{-3}(x).$$

Using the change of variable (3.2), we get

$$(\Phi_x h)_i \;=\; \frac{3\eta}{2a\varepsilon}\left(\tilde{v} + \omega\lambda_i p_i^\perp + 2\varepsilon\omega r^\perp - \frac{3}{4}a\varepsilon\sum_{j,j\neq i}\frac{1}{\rho_{i,j}}\left\{(\tilde{v} + \omega\lambda_j p_j^\perp) + (e_{i,j}\cdot(\tilde{v} + \omega l_j p_j^\perp))e_{i,j}\right\}\right)$$
$$+ \varepsilon\alpha_{\varepsilon,\lambda,\theta}(\tilde{v}, \omega),$$

$$(\Phi_x k)_i \;=\; \frac{3\eta}{2a\varepsilon}\left(\dot{\lambda}_i p_i - \frac{3}{4}a\varepsilon\sum_{j,j\neq i}\frac{\dot{\lambda}_j}{\rho_{i,j}}(p_j + (e_{i,j}\cdot p_j)e_{i,j})\right) + \varepsilon\beta_{\varepsilon,\lambda,\theta}\dot{l},$$

with as $\varepsilon \downarrow 0$,

$$\|\alpha_{x,\lambda,\theta}\|_{\mathcal{H}^{-1/2}},\; \|\nabla_\lambda\alpha_{x,\lambda,\theta}\|_{\mathcal{H}^{-1/2}},\; \|\beta_{x,\lambda,\theta}\|_{\mathcal{H}^{-1/2}},\; \|\nabla_\lambda\beta_{x,\lambda,\theta}\|_{\mathcal{H}^{-1/2}} \;=\; O(1).$$

Plugging these formulas in (3.3), we obtain the conditions

$$3\tilde{v} + \left(\sum_{i=1}^3 \lambda_i p_i^\perp\right)\omega - \frac{3}{4}a\varepsilon\left(\sum_{i,j,\,i\neq j}\frac{1}{\rho_{i,j}}(Id + e_{i,j}\otimes e_{i,j})\right)\tilde{v} - \frac{3}{4}a\varepsilon\left(\sum_{i,j,\,i\neq j}\frac{1}{\rho_{i,j}}\lambda_j(p_j^\perp + (p_j^\perp\cdot e_{i,j})e_{i,j})\right)\omega$$

$$+ \varepsilon^2\alpha^f_{\varepsilon,\lambda,\theta}(\tilde{v}, \omega) \;=\; -\sum_{i=1}^3\dot{\lambda}_i p_i + \frac{3}{4}a\varepsilon\sum_{i,j,\,i\neq j}\frac{\dot{\lambda}_j}{\rho_{i,j}}(p_j + (p_j\cdot e_{i,j})e_{i,j}) + \varepsilon^2\beta^f_{\varepsilon,\lambda,\theta}\dot{\lambda}, \qquad (3.4)$$

$$\left(\sum_{i=1}^3 \lambda_i p_i^\perp\right)\tilde{v} + \left(\sum_{i=1}^3 \lambda_i^2\right)\omega + \varepsilon\alpha^t_{\varepsilon,\lambda,\theta}(\tilde{v}, \omega) \;=\; \varepsilon\beta^t_{\varepsilon,\lambda,\theta}\dot{\lambda}, \qquad (3.5)$$

with $\quad\alpha^f_{x,\lambda,\theta},\; \alpha^t_{x,\lambda,\theta},\; \beta^f_{x,\lambda,\theta},\; \beta^t_{x,\lambda,\theta},\; =\; O(1)\quad$ and $\quad\nabla_\lambda\alpha^f_{x,\lambda,\theta},\; \nabla_\lambda\alpha^t_{x,\lambda,\theta},\; \nabla_\lambda\beta^f_{x,\lambda,\theta},\; \nabla_\lambda\beta^t_{x,\lambda,\theta}\; =\; O(1).$

To obtain (3.4) and (3.5) we used the identities $p_i \cdot p_i = 1$, $p_i^\perp \cdot p_i = 0$ and $\int_{\partial B} r = 0$. Note that in (3.5) we only keep the leading order terms: it is not necessary to keep track of $O(\varepsilon)$ terms in the computations below.

Proof of (1.7)

We have seen in the introduction that, by linearity, the global displacement takes the form (1.4) which reads here

$$\tilde{v} \;=\; \tilde{V}_{\lambda,c,\theta}\dot{\lambda}, \qquad \omega \;=\; \Omega_{\lambda,c,\theta}\dot{\lambda}, \qquad (3.6)$$

In the special case $\lambda = \overline{\lambda} := (1,1,1)$, the problem is invariant by the three reflections with respect to $\mathcal{D}_1, \mathcal{D}_2$ and $\mathcal{D}_3$. As a consequence, we necessarily have $\omega = 0$ and $\tilde{v} = \kappa \sum_i \dot{\lambda}_i p_i$ for some $\kappa \in \mathbf{R}$. Solving (3.4), we get

$$\tilde{v} = (1 + O(\varepsilon)) \left\{ -\frac{1}{3} \sum_{i=1}^{3} \dot{\lambda}_i p_i \right\}, \qquad \omega = 0.$$

From this we obtain, for $i = 1, 2, 3$,

$$F^i_{\frac{\lambda}{\varepsilon}, c, \theta} = \left(e_i, -\frac{p_i}{3} + O(\varepsilon), 0 \right). \tag{3.7}$$

The next step is to compute the brackets $\left\{ (F_X^k \cdot \nabla) T_X^l - (F_X^l \cdot \nabla) T_X^k \right\}_{|_{\lambda = \overline{\lambda}}}$ for $k \neq l$. Let us fix $1 \leq k, l \leq 3$ with $k \neq l$. By translation invariance T_X^l does not depend on c, so, taking into account (3.7), we have

$$\left\{ (F_X^k \cdot \nabla) T_X^l \right\}_{|_{\lambda = \overline{\lambda}}} = \varepsilon \frac{\partial}{\partial \lambda_k} T_X^l{}_{|_{\lambda = \overline{\lambda}}} = \frac{\partial}{\partial \lambda_k} (\tilde{v}, \varepsilon \omega)_{|_{\lambda = \overline{\lambda}}} =: (\tilde{V}, \varepsilon W), \tag{3.8}$$

where $\tilde{v}, \omega$ solve (3.4)(3.5). Differentiating (3.4) with respect to λ_k and using (3.7) we obtain

$$(3Id + O(\varepsilon)) \tilde{V}$$

$$= \frac{3}{4} a\varepsilon \frac{\partial}{\partial \lambda_k} \left\{ \sum_{i,j, i \neq j} \frac{1}{\rho_{i,j}} (Id + e_{i,j} \otimes e_{i,j}) \right\} \left(-\frac{p_l}{3} \right) + \frac{3}{4} a\varepsilon \frac{\partial}{\partial \lambda_k} \left\{ \sum_{j, j \neq l} \frac{1}{\rho_{l,j}} (p_l + (p_l \cdot e_{l,j}) e_{l,j}) \right\} + O(\varepsilon^2). \tag{3.9}$$

In particular, we used the identities $\sum_{i=1}^{3} \lambda_i p_i^\perp = \sum_{i \neq j} \lambda_j p_j^\perp = \sum_{i \neq j} \lambda_j (p_j^\perp \cdot e_{i,j}) e_{i,j} = 0$ for $\lambda = \overline{\lambda}$.

Since the unknown W does not appear in this system, we can solve it and obtain $\tilde{V}$ at leading order. After straightforward calculations (postponed to the Appendix below) we conclude:

$$\tilde{V} = a\varepsilon \frac{\sqrt{3}}{2^4 3^2} (-9 p_l - 19 p_k) + O(\varepsilon^2). \tag{3.10}$$

Similarly, differentiating (3.5) with respect to λ_k, we easily obtain

$$W = \frac{1}{9} p_k^\perp \cdot p_l + O(\varepsilon). \tag{3.11}$$

Finally (3.10)(3.11) together with (3.8) yield

$$\left\{ (F_X^k \cdot \nabla) T_X^l - (F_X^l \cdot \nabla) T_X^k \right\}_{|_{\lambda = \overline{\lambda}}} = \varepsilon \left(a \frac{5\sqrt{3}}{2^3 3^2} (p_l - p_k), \frac{2}{9} p_k^\perp \cdot p_l \right) + O(\varepsilon^2)$$

and

$$\det \left((F_X^1 \cdot \nabla) T_X^2 - (F_X^2 \cdot \nabla) T_X^1, \ (F_X^2 \cdot \nabla) T_X^3 - (F_X^3 \cdot \nabla) T_X^2, \ (F_X^3 \cdot \nabla) T_X^1 - (F_X^1 \cdot \nabla) T_X^3 \right)_{|_{X = X_\varepsilon}}$$

$$= -\frac{5^2 a^2 \varepsilon^3}{2^5 3^5} \det \begin{pmatrix} p_1 - p_2 & p_2 - p_3 & p_3 - p_1 \\ p_1^\perp \cdot p_2 & p_2^\perp \cdot p_3 & p_3^\perp \cdot p_1 \end{pmatrix} + O(\varepsilon^4) = -\frac{5^2 a^2 \varepsilon^3}{2^7 3^3} + O(\varepsilon^4).$$

This implies that the determinant does not vanish for ε small enough, ending the proof of (1.7) and Theorem 1.1.

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A. Appendix

Starting from (3.9) we exhibit here the computations leading to (3.10). We have to compute for $\lambda = \overline{\lambda}$ the quantities

$$I := \frac{\partial}{\partial \lambda_k} \left\{ \sum_{i,j,\, i \neq j} \frac{1}{\rho_{i,j}} (Id + e_{i,j} \otimes e_{i,j}) \right\} \left(-\frac{p_l}{3} \right) \quad \text{and} \quad II := \frac{\partial}{\partial \lambda_k} \left\{ \sum_{j,j \neq l} \frac{1}{\rho_{l,j}} (p_l + (p_l \cdot e_{l,j}) e_{l,j}) \right\}.$$

We start by some intermediate identities. Recall that by definition, $\rho_{i,j} = |\lambda_i p_i - \lambda_j p_j|$ and since $p_i = -\frac{1}{2} p_j \pm \frac{\sqrt{3}}{2} p_j^\perp$, we have $\rho_{i,j} = \sqrt{(\lambda_j + \frac{\lambda_i}{2})^2 + \frac{3}{4}\lambda_i^2}$. So for $\lambda = \overline{\lambda}$,

$$\rho_{i,j} = \sqrt{3}, \qquad \frac{\partial}{\partial \lambda_k} \rho_{i,j} = 0 \quad \text{if} \quad i \neq j \neq k, \qquad \frac{\partial}{\partial \lambda_k} \rho_{i,j} = \frac{\sqrt{3}}{2} \quad \text{if} \quad i = k \quad \text{or} \quad j = k.$$

Next by definition, we have for $i \neq j$, $e_{i,j} = \frac{1}{\rho_{i,j}}(\lambda_i p_i - \lambda_j p_j)$. Thus for $\lambda = \overline{\lambda}$,

$$\frac{\partial}{\partial \lambda_k} e_{k,j} = \frac{1}{2\sqrt{3}}(p_j + p_k), \qquad \text{leading to} \qquad \frac{\partial}{\partial \lambda_k}(e_{k,j} \otimes e_{k,j}) = \frac{1}{3}(p_k \otimes p_k - p_j \otimes p_j).$$

We will also need the equalities

$$p_l \cdot e_{k,j} = 0 \quad \text{if} \quad j \neq l, \qquad p_l \cdot e_{k,l} = -\frac{\sqrt{3}}{2}, \qquad p_1 + p_2 + p_3 = 0.$$

Using these identities we compute

$$I = 2\frac{\partial}{\partial \lambda_k}\left\{\sum_{j,\,j\neq k}\frac{1}{\rho_{k,j}}(Id + e_{k,j}\otimes e_{k,j})\right\}\left(-\frac{p_l}{3}\right) = \frac{\sqrt{3}}{3^2 2}(-p_l - 7p_k), \tag{A.1}$$

$$II = \frac{\partial}{\partial \lambda_k}\left\{\frac{1}{\rho_{k,l}}(p_l + (e_{k,l}\otimes e_{k,l})p_l\right\} = \frac{\sqrt{3}}{3^2 2^2}(-7p_l - 5p_k). \tag{A.2}$$

Finally, plugging (A.1) and (A.2) in (3.9), we get (3.10).