MICRO-MACRO MODELLING OF AN ARRAY OF SPHERES INTERACTING THROUGH LUBRICATION FORCES

A. LEFEBVRE-LEPOT AND B. MAURY

Abstract. We consider here a discrete system of spheres interacting through a lubrication force. This force is dissipative, and singular near contact: it behaves like the reciprocal of the interparticle distance. We propose a macroscopic constitutive equation which is built as the natural continuous counterpart of this microscopic lubrication model. This model, which is of the newtonian type, relies on an extensional viscosity, which is proportional to the reciprocal of the local fluid fraction. We then establish the convergence in a weak sense of solutions to the discrete problem towards the solution to the partial differential equation which we identified as the macroscopic constitutive equation.

Keywords: Lubrication theory; fluid-solid interaction; homogeneization.

AMS Subject Classification: 35J25, 74Q05, 74Q15, 76M50

1. Introduction

We are interested in the macroscopic behaviour of highly dense suspensions of rigid spheres. The case of dilute suspensions has motivated a huge litterature since 1906, where the first derivation of the first order expansion of the equivalent viscosity with respect to the solid fraction has been proposed by Einstein (see [10], or [13, §22], where Einstein's approach is detailed). This approach was extended to semi-dilute suspensions (see [2]) which leads to second order asymptotic expansions of the apparent viscosity with respect to the solid fraction. For intermediate volume fractions (say, between 5 and 30 %), particles interact in a complex way, so that direct numerical simulation is usually considered as the only way to account for the complex phenomena which are likely to occur at the microscopic level (see e.g. [12], or [14] for an illustration of how direct simulation can be used to investigate the apparent viscosity of a suspension of particles at intermediate volume fraction).

For highly packed suspensions in a viscous fluid, interparticle distances tend to approach zero, so that lubrication forces between particles in quasi-contact become predominant. It is then natural to consider the suspension as a collection of spheres, each of which interacting with its close neighbours only, according to some model accounting for the lubrication forces.

Address: A. Lefebvre-Lepot, CMAP, CNRS et Ecole Polytechnique, route de Saclay, 91128 Palaiseau Cedex, France
B. Maury, Laboratoire de Mathématiques d'Orsay, Université Paris-Sud 11, 91405 Orsay Cedex, France
Aline.Lefebvre@cmap.polytechnique.fr, Bertrand.Maury@math.u-psud.fr

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In Ref. [17], a first attempt was proposed to investigate the behaviour of the apparent shear viscosity of a suspension in the neighbourhood of the maximal packing solid fraction $\Phi_{\max}$. A model is proposed, which gives a shear viscosity which is of the order $(1-\Phi/\Phi_{\max})^{-2}$ where Φ is the solid fraction. In Ref. [7], the authors investigate the asymptotic behaviour, as ε goes to 0, of a set of particles under the assumption that distances between neighbouring particles are subject to behave like ε. In this framework, the authors establish that the apparent shear viscosity behaves like $1/\varepsilon^{3/2}$. This approach extents a previous work [11] where periodic arrays of spheres are considered. In this context, the elongational viscosity can be shown to behave like $1/d$ where d is the constant distance between neighbouring spheres.

The approach we propose here is based on a simpler model from the geometric standpoint, as the spheres are supposed to be aligned. On the other hand it generalizes these works in the sense that no assumption is made on the distances: the macroscopic behaviour depends on the local solid fraction only. Contacts between neighboring particles are even allowed, and a special attention has been paid to the way we express the continuous model so that macroscopic clusters can be taken into account (the local viscosity within a cluster is infinite).

This work can be related to classical results in homogeneization of elliptic operators (see e.g. [18, 19] and references therein). It consists in studying the asymptotic behaviour of a sequence of solutions u_ε to elliptic problems:

$$-\mathrm{div}(a_\varepsilon(x)\nabla u_\varepsilon) = f \text{ in } \Omega$$

$$u_\varepsilon = 0 \text{ on } \partial\Omega$$

where Ω is a bounded set of $\mathbb{R}^d$ and a_ε a sequence of equi-coercive matrix-valued functions. Most of the results concerning the asymptotic behaviour of such systems suppose that the sequence a_ε is bounded in L^∞. In the two dimensional case, some results were obtained in [4, 5] under more general assymptions. However, a_ε still have to belong to L^∞. In dimension one, when the diffusion coefficient a_ε belongs, together with its reciprocal, to $L^\infty(\Omega)$, the closure of the set defined by such systems has been studied in [1]. In our model, such results do not hold anymore. Indeed, since we allow contacts between particles, one of the difficulties of this work is that a_ε can take infinite values on non negligible subsets of Ω. Note that another main difference between our work and classical homogenization results stands in the fact that we study non stationnary problems: the density evolves according to a transport equation.

We prove in Section 3 that the limit elongational viscosity behaves singularly with respects to the vanishing fluid fraction. This approach leads to an equation of the elliptic type

$$-\partial_x\left(\frac{1}{1-\rho}\partial_x u\right) = \rho f. \tag{1}$$

where u is the velocity field, ρ the solid fraction (which is 1 when all particles are in contact), and f an external body force. This result is extended to moving particle collections in Section 4. In this latter situation, we establish convergence to non-stationnary velocity and density fields, such that instantaneous balance of forces reads as previously, and solid phase motion is described by the standard transport equation

$$\partial_t \rho + \partial_x(\rho u) = 0. \tag{2}$$

2. Discrete Model

Consider (see Fig. 1) two rigid spheres imbedded in a viscous fluid, subject to move horizontally. Denoting by q_1 and q_2 the abscisses of their centers, by u_1 and u_2 their

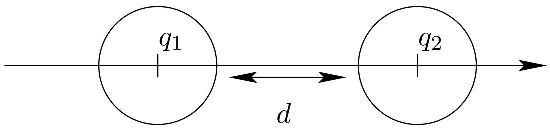

Figure 1. Lubrication model

instantaneous velocities and by d the border-to-border distance, the leading term in the asymptotic expansion of the interaction force is (see e.g. [6]):

$$F_{1\to 2} = -\kappa \frac{u_2 - u_1}{d}, \tag{3}$$

where κ is a constant which depends on the viscosity of the lubricating fluid and the radii of the spheres. We shall take $\kappa = 1$ in what follows. Consider now (see Fig. 2) an array of $N+1$ spheres, on the x-axis, with the same radius ε. We set the first and the last sphere at positions 0 and 1, respectively. As a consequence, the number of degrees of freedom is $N-1$, whereas the number of actual spheres is $N+1$.

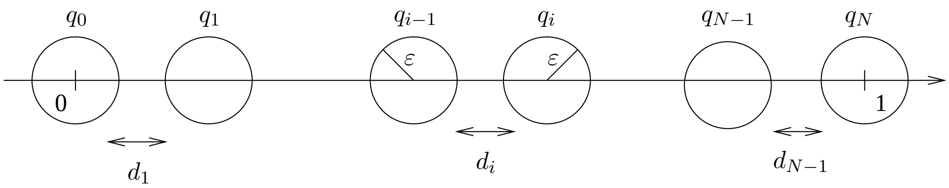

Figure 2. Geometry

Definition 2.1. Given a vector of positions $\mathbf{q} = (q_i)_{1\le i\le N-1}$, we say that $\mathbf{q}$ is ε-feasible (spheres do not overlap), if

$$q_i - q_{i-1} - 2\varepsilon \ge 0 \quad \forall i = 1\dots N, \text{ with } q_0 = 0 \text{ and } q_N = 1$$

and strictly ε-feasible if all inequalities are strict (spheres do not touch).

We denote by $d_i = q_i - q_{i-1} - 2\varepsilon$ the distance between spheres i and $i-1$, by u_i the instantaneous velocity of sphere i, and by $\mathbf{u} = (u_i)_{1\le i\le N-1}$ the velocity vector. Velocities of the extremal spheres 0 and N are taken as 0 (see remark 2.4 for non-zero extremal velocities). Given a strictly ε-feasible vector $\mathbf{q}$, we define $A(\mathbf{q})$ as the $(N-1)\times(N-1)$ tridiagonal stiffness matrix

$$(4)\quad A(\mathbf{q}) = \begin{pmatrix} \frac{1}{d_1}+\frac{1}{d_2} & -\frac{1}{d_2} & & & & & \\ -\frac{1}{d_2} & \frac{1}{d_2}+\frac{1}{d_3} & -\frac{1}{d_3} & & & & \\ & \ddots & \ddots & \ddots & & & \\ & & -\frac{1}{d_i} & \frac{1}{d_i}+\frac{1}{d_{i+1}} & -\frac{1}{d_{i+1}} & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & -\frac{1}{d_{N-2}} & \frac{1}{d_{N-2}}+\frac{1}{d_{N-1}} & -\frac{1}{d_{N-2}} \\ & & & & & -\frac{1}{d_{N-1}} & \frac{1}{d_{N-1}}+\frac{1}{d_N} \end{pmatrix}$$

with $d_i = q_i - q_{i-1} - 2\varepsilon$. Consider now a set of forces f_1, f_2,… f_{N-1}, and the corresponding vector $\mathbf{f}$. From (3), the balance of forces reads

$$(5)\qquad -A(\mathbf{q})\mathbf{u} + \mathbf{f} = 0.$$

Proposition 2.2. *Given a strictly ε-feasible vector $\mathbf{q} \in \mathbb{R}^{N-1}$, a force field $\mathbf{f} \in \mathbb{R}^{N-1}$, problem (5) has a unique solution $\mathbf{u}$, and we shall write*

$$\mathbf{u} = (u_i)_{1\le i\le N-1} = \mathcal{P}(\mathbf{q}, \mathbf{f}, \varepsilon).$$

This solution can be written

$$(6)\qquad u_i = \frac{1}{D_N}\left\{(D_N - D_i)\sum_{k=1}^{i} D_k f_k + D_i \sum_{k=i+1}^{N-1}(D_N - D_k) f_k\right\} \quad \forall i = 1\dots N-1$$

with $D_i = \sum_{j=1}^{i} d_j$.

Proof. Matrix A, which is similar to the matrix obtained by discretizing the Laplace operator with Dirichlet boundary condition by finite differences, is symmetric positive definite, and the vector $\mathbf{u}$ is immediately checked to solve the system. □

This approach can be extended to ε-feasible situations in a large sense (particles are allowed to get into contact). As the interaction force (which tends to penalize the relative velocity) blows up when particles tend to get into contact, we simply consider that two particles in contact have the same velocity. This situation can be formalized the following way: The $N+1$ particules form N_c clusters, and the k-th cluster contains the N_k+1 particles i_k, i_k+1, …, i_k+N_k (see Fig. 3). The balance of forces now reads

$$\forall i \notin \cup_k [i_k, i_k + N_k], \qquad \frac{u_{i+1} - u_i}{d_{i+1}} - \frac{u_i - u_{i-1}}{d_i} = -f_i, \tag{7}$$

$$\forall k \in [1, N_c], \quad \left\{ \begin{array}{l} u_{i_k} = u_{i_{k+1}} = \ldots = u_{i_k + N_k}, \\ \dfrac{u_{i_k+N_k+1} - u_{i_k+N_k}}{d_{i_k+N_k+1}} - \dfrac{u_{i_k} - u_{i_k-1}}{d_{i_k}} = -\displaystyle\sum_{i=i_k}^{i_k+N_k} f_i. \end{array} \right. \tag{8}$$

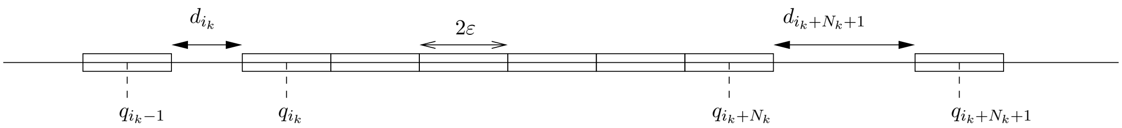

FIGURE 3. Non-strictly ε-feasible configuration

Proposition 2.3. *Given an ε-feasible vector $\mathbf{q} \in \mathbb{R}^{N-1}$, a force field $\mathbf{f} \in \mathbb{R}^{N-1}$, problem (7-8) has a unique solution, and we shall write as before*

$$\mathbf{u} = (u_i)_{1 \le i \le N-1} = \mathcal{P}(\mathbf{q}, \mathbf{f}, \varepsilon).$$

An explicit expression of this solution is given by (6).

Remark 2.4. It is possible to set extremal velocities u_0 and u_N to non-zero values: in that case, the balance of forces is given by

$$A(\mathbf{q})\mathbf{u} = \mathbf{f} + \mathbf{b},$$

where $\mathbf{b}$ contains the non homogeneous dirichlet boundary conditions:

$$\mathbf{b} = (u_0/d_1, 0, \ldots, 0, u_N/d_N)^t.$$

The extension of Proposition 2.3 to that case is straightforward.

3. Asymptotic Behaviour of the Discrete Solutions

Let I denote $]0,1[$. Firstly, we build a new operator $\tilde{\mathcal{P}}$, which is our key tool to connect the microscopic level to the macroscopic one. This operator is defined the following way: Given $\varepsilon > 0$, an ε-feasible position vector $\mathbf{q} \in \mathbb{R}^{N-1}$ (as stated by Definition 2.1, it represents a distribution of $N+1$ particles whose centers are located at $q_0 = 0$, q_i for $i = 1, \ldots, N-1$ and $q_N = 1$, with common radius ε), a force density $f \in L^1(I)$, we define $\mathbf{u}$ as $\mathcal{P}(\mathbf{q}, \mathbf{f}^\varepsilon, \varepsilon)$ (see Proposition 2.2 or 2.3, depending on whether $\mathbf{q}$ is feasible or strictly feasible), where $\mathbf{f}^\varepsilon$ is defined by

$$f_i^\varepsilon = \frac{1}{2\varepsilon} \int_{q_i - \varepsilon}^{q_i + \varepsilon} f(s) ds \text{ for } 1 \le i \le N-1.$$

Now, to this vector $\mathbf{u} = \mathcal{P}(\mathbf{q}, \mathbf{f}^\varepsilon, \varepsilon) = (u_i)_{i=1\ldots N-1}$, we associate a piecewise affine function u defined by

$$\begin{array}{c} u \in C^0(\overline{I}) \ , \quad u \text{ affine on } [q_i, q_{i+1}] \quad \forall i = 0, \ldots, N-1 \\ u(q_i) = u_i \quad \forall i = 1, \ldots, N-1 \ , \quad u(0) = u(1) = 0. \end{array} \tag{9}$$

We shall write $u = \tilde{\mathcal{P}}(\mathbf{q}, f, \varepsilon)$ (see Fig. 4).

In the same spirit, for any $\varepsilon > 0$, and any ε-feasible position vector $\mathbf{q} \in \mathbb{R}^{N-1}$, we define $\chi(\mathbf{q}, \varepsilon)$ as the characteristic function of the solid phase associated to the $(\mathbf{q}, \varepsilon)$ distribution:

$$\chi(\mathbf{q}, \varepsilon) = \sum_{i=1}^{N-1} \mathbf{1}_{]q_i-\varepsilon, q_i+\varepsilon[} + \mathbf{1}_{]0,\varepsilon[} + \mathbf{1}_{]1-\varepsilon,1[}. \tag{10}$$

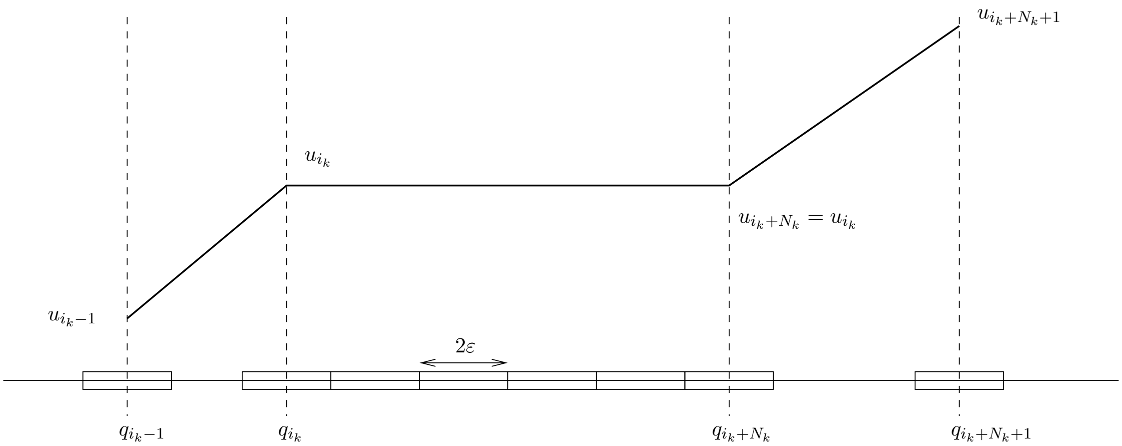

FIGURE 4. Definition of $u = \tilde{\mathcal{P}}(\mathbf{q}, f, \varepsilon)$

Before we state the main convergence theorem, we still have to give a sense to (1) when the density $\rho \in [0, 1]$ is allowed to take value 1, even on a set of non-zero measure.

Proposition 3.1. *Let $K : I \mapsto \mathbb{R}^+ \cup \{+\infty\}$ be measurable, $K(x) \geq \alpha > 0$ for almost every $x \in I$, $\varphi \in H^{-1}(I)$, and let J be defined as*

$$v \in H_0^1(I) \longmapsto J(v) = \int_I K(x)|\partial_x v|^2 - < \varphi, v > \in \mathbb{R} \cup \{+\infty\}.$$

There exists a unique $u \in H_0^1(I)$ which realizes the minimum of J over $H_0^1(I)$. If there exists $f \in L^1(I)$ such that $< \varphi, v >= \int fv$, we shall say that u is a generalized solution to

$$-\partial_x (K(x)\partial_x u) = f.$$

Proof. The functional J is convex (strictly convex over its domain), coercive, and it can be written

$$J(v) = \sup_{n\in\mathbb{N}} \left(\int_I \min(K(x), n)|\partial_x v|^2 - \int_I fv \right),$$

thus it is l.s.c. as a supremum of a family of l.s.c functions. Therefore it admits a unique minimizer. Note that the minimization problem is equivalent to the problem which consists in minimizing the same functional J over the set

$$H_K = \left\{ v \in H_0^1(I), \partial_x v = 0 \text{ a.e. in } D(K)^c \text{ and } \int_{D(K)} K(x)|\partial_x v|^2 < \infty \right\}. \tag{11}$$

where $D(K) = \{x \in I, K(x) < +\infty\}$ is the domain of K. Consequently, u is characterized by the variational formulation

$$u \in H_K \ , \quad \int_{D(K)} K(x)\partial_x u \, \partial_x v = \int_I f v \quad \forall v \in H_K. \tag{12}$$

□

We may now state the convergence result.

Theorem 3.2. *Let $f \in L^\infty(I)$ be a Lipschitz force density, and $\rho \in L^\infty(I)$ a solid fraction, with $\rho(x) \in [0,1]$ a.e. in I. Let $(\mathbf{q}^\varepsilon)_\varepsilon$ be a sequence of ε-feasible position vectors (see Definition 2.1), $\mathbf{q}^\varepsilon \in \mathbb{R}^{N^\varepsilon - 1}$ with $N^\varepsilon = 1/\varepsilon$. We introduce $\chi^\varepsilon = \chi(\mathbf{q}^\varepsilon, \varepsilon)$ (see (10)), and we assume that χ^ε converges toward ρ in $L^\infty(I)$ weak-$\star$.*

Then $u^\varepsilon = \tilde{\mathcal{P}}(\mathbf{q}^\varepsilon, f, \varepsilon)$ solution to the discrete model (see (9)) converges weakly toward u in $H_0^1(I)$ as ε converges to 0, where u is the solution to

$$-\partial_x \left(\frac{1}{1-\rho} \partial_x u \right) = \rho f, \tag{13}$$

in the sense of Proposition 3.1 (i.e. characterized by (12)).

Remark 3.3. For any $\rho \in L^\infty(I)$ with $\rho(x) \in [0,1]$ a.e. in I, one can easily construct a sequence $(\mathbf{q}^\varepsilon)_\varepsilon$, ε-feasible, such that $\chi^\varepsilon \xrightarrow{\star} \rho$ in $L^\infty(I)$. Let ε be equal to $1/N$. We define $q_0^\varepsilon = 0$. Then, we denote by l_1^ε the real such that $\int_{q_0^\varepsilon}^{l_1^\varepsilon} \rho = \varepsilon$. Since $\rho \leq 1$ we have $l_1^\varepsilon \geq q_0^\varepsilon + \varepsilon$ and we can define $q_1^\varepsilon = l_1^\varepsilon + \varepsilon$. Similarly, to define q_1^ε, we first chose l_2^ε such that $\int_{l_0^\varepsilon}^{l_1^\varepsilon} \rho = 2\varepsilon$ and define $q_2^\varepsilon = l_2^\varepsilon + \varepsilon$. We iterate to obtain $(\mathbf{q}^\varepsilon)_\varepsilon$, ε-feasible. Moreover we have $\int_{l_i^\varepsilon}^{l_{i+1}^\varepsilon} (\chi^\varepsilon - \rho) = 0$ and it follows that $\chi^\varepsilon \xrightarrow{\star} \rho$ in $L^\infty(I)$ when ε goes to zero (see the proof of lemma 3.6, where a similar property is established).

Proof. The proof is based on some technical lemmas. For readability reasons, we postpone the proofs of the lemmas to the end of the section.

As a first step, we define ρ^ε, piecewise constant, as the proportion of solid on each subinterval $[q_{i-1}^\varepsilon, q_i^\varepsilon]$ in the following way: let d_i^ε be the distance between particles $i-1$ and i, then

$$\forall i = 1, \ldots, N^\varepsilon \ , \quad \rho^\varepsilon = \rho_i^\varepsilon = 1 - \frac{d_i^\varepsilon}{q_i^\varepsilon - q_{i-1}^\varepsilon} \text{ on } [q_{i-1}^\varepsilon, q_i^\varepsilon]. \tag{14}$$

where $q_0^\varepsilon = 0$ and $q_{N^\varepsilon}^\varepsilon = 1$. This definition is illustrated by Fig. 5. Note that, in this figure and in the following of the proof, we shall drop the superscript ε for indexes concerning the clusters in order to alleviate notations (so that, for instance, we shall write N_c for N_c^ε, $q_{i_k}^\varepsilon$ for $q_{i_k^\varepsilon}^\varepsilon$ or $q_{i_k+N_k}^\varepsilon$ for $q_{i_k^\varepsilon+N_k^\varepsilon}^\varepsilon$).

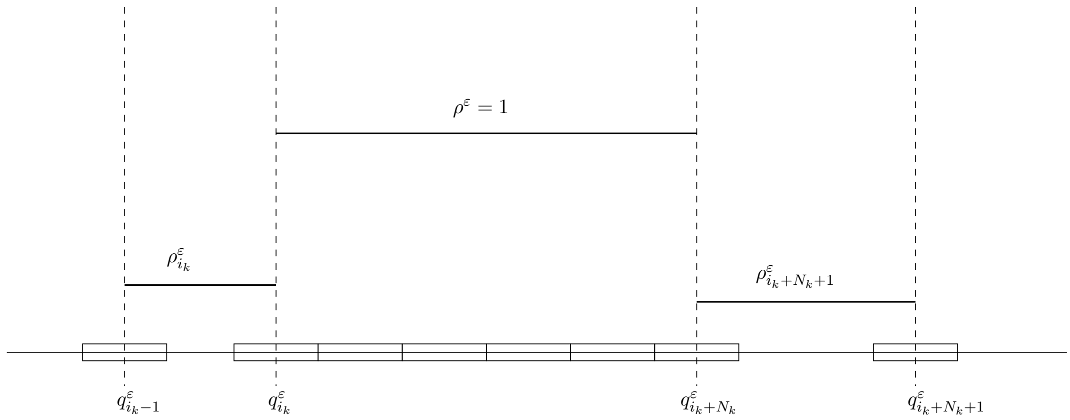

FIGURE 5. Definition of ρ^ε

Define now w^ε (see Fig. 6), affine by part, as the discrete counterpart of $\partial_x u/(1-\rho)$ given by

$$\left\{ \begin{array}{ll} w^\varepsilon(q_i^\varepsilon - \varepsilon) = w_i^\varepsilon & \text{for } i = 1 \ldots N^\varepsilon - 1 \\ w^\varepsilon(q_i^\varepsilon + \varepsilon) = w_{i+1}^\varepsilon & \end{array} \right. \tag{15}$$

with

$$\left\{ \begin{array}{ll} w_i^\varepsilon = \dfrac{1}{1-\rho_i^\varepsilon}\partial_x u^\varepsilon & \text{, if } \rho_i^\varepsilon < 1 \\ w_i^\varepsilon = \beta_i^\varepsilon & \text{, if } \rho_i^\varepsilon = 1 \end{array} \right. \tag{16}$$

where $(\beta_i^\varepsilon)_{i_k+1 \le i \le i_k+N_k}$ corresponding to the kth cluster, is the solution to the following system:

$$\left\{ \begin{array}{ccccc} \beta^\varepsilon_{i_k+1} & - & \dfrac{u^\varepsilon_{i_k} - u^\varepsilon_{i_k-1}}{d^\varepsilon_{i_k}} & = & -2\varepsilon f^\varepsilon_{i_k}, \\ \beta^\varepsilon_{i_k+2} & - & \beta^\varepsilon_{i_k+1} & = & -2\varepsilon f^\varepsilon_{i_k+1}, \\ & & & \vdots & \\ \beta^\varepsilon_{i_k+N_k} & - & \beta^\varepsilon_{i_k+N_k-1} & = & -2\varepsilon f^\varepsilon_{i_k+N_k-1}, \\ \dfrac{u^\varepsilon_{i_k+N_k+1} - u^\varepsilon_{i_k+N_k}}{d^\varepsilon_{i_k+N_k+1}} & - & \beta^\varepsilon_{i_k+N_k} & = & -2\varepsilon f^\varepsilon_{i_k+N_k}. \end{array} \right. \tag{17}$$

Note that, summing up all equations of (17) we recognize the balance of forces on the kth cluster given by (8).

Remark 3.4. The idea behind the above construction is that β_i can be seen as the cohesion force between particles $i-1$ and i. A first way to notice it is to note that $\mathbf{u}^\varepsilon$ is the limit of $\mathbf{u}^{\varepsilon,\eta}$ where $\mathbf{u}^{\varepsilon,\eta}$ is the solution to system (5) with $d_i = \eta > 0$ for i between i_k+1 and i_k+N_k and that we have

$$\forall k\ , \quad \forall j \in [1, N_k]\ , \quad \beta^\varepsilon_{i_k+j} = \lim_{\eta \to 0} \frac{u^{\varepsilon,\eta}_{i_k+j} - u^{\varepsilon,\eta}_{i_k+j-1}}{\eta}.$$

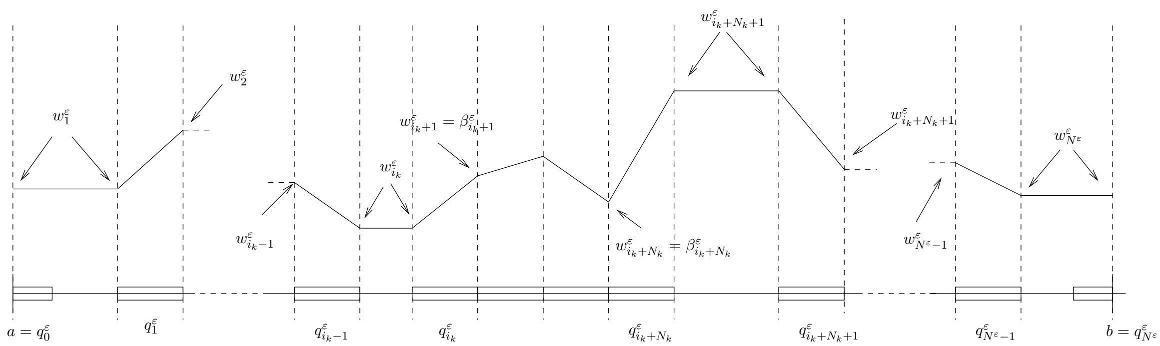

FIGURE 6. Definition of w^ε

Another way to define these cohesion forces is to consider the following minimization problem: minimize

$$J(\mathbf{v}) = \sum_{i \notin \cup_k [i_k+1, i_k+N_k]} \frac{1}{2} \frac{(v_i - v_{i-1})^2}{d_i^\varepsilon} + \sum_{i=1}^{N-1} f_i^\varepsilon v_i$$

over $K = \{\mathbf{v} \ , \ \forall i \in \cup_k [i_k+1, i_k+N_k],\ v_i = v_{i-1}\}$. This problem is equivalent to problem (7-8) and β_i^ε turns up as the Lagrange multiplier associated to the constraint $v_i = v_{i-1}$.

The next lemma shows that (13) is true at the discrete level:

Lemma 3.5. *For any $\varepsilon > 0$,*

$$-\partial_x (w^\varepsilon) = f^\varepsilon \tag{18}$$

where $f^\varepsilon = \displaystyle\sum_{i=1}^{N^\varepsilon-1} f_i^\varepsilon \mathbf{1}_{]q_i^\varepsilon - \varepsilon, q_i^\varepsilon + \varepsilon[}$.

The idea of the proof is now to let ε go to 0 in (18). To that purpose, we first study ρ^ε and $2\varepsilon f^\varepsilon$ when ε tends to zero:

Lemma 3.6. $\rho^\varepsilon \overset{\star}{\rightharpoonup} \rho$ *in* $L^\infty(I)$.

Lemma 3.7. $f^\varepsilon \overset{\star}{\rightharpoonup} \rho f$ *in* $L^\infty(I)$.

To study the convergence of $(u^\varepsilon)_\varepsilon$, we prove in next lemma that it is bounded in $H_0^1(I)$.

Lemma 3.8. $(u^\varepsilon)_\varepsilon$ *is bounded in* $H_0^1(I)$.

Hence, we can extract a subsequence of $(u^\varepsilon)_\varepsilon$ (still denoted by $(u^\varepsilon)_\varepsilon$) such that

$$u^\varepsilon \rightharpoonup u \text{ in } H_0^1(I).$$

In order to pass to the limit in the left-hand side of (18), we are going to study the convergence of $(w^\varepsilon)_\varepsilon$. It will follow from the next lemma.

Lemma 3.9. *$(w^\varepsilon)_\varepsilon$ and $(\partial_x w^\varepsilon)_\varepsilon$ are bounded in $L^\infty(I)$.*

Consequently, from Ascoli theorem, we can find w in $\mathcal{C}^0(I)$ and a subsequence of $(w^\varepsilon)_\varepsilon$ (still denoted by $(w^\varepsilon)_\varepsilon$) such that

$$w^\varepsilon \to w \text{ in } \mathcal{C}^0(I). \tag{19}$$

Then, convergence of w^ε and ρ^ε make it possible to establish the following lemma:

Lemma 3.10. *$\partial_x u^\varepsilon \xrightarrow{\star} (1-\rho)w$ in $L^\infty(I)$.*

We now come to the last step of the proof: let ε tend to zero in (18) and obtain asymptotically

$$-\partial_x\left(\frac{1}{1-\rho}\partial_x u\right) = \rho f,$$

in the sense of Proposition 3.1, characterized by (12).

First, Lemma 3.10 implies $\partial_x u = (1-\rho)w$, so $u \in H_{1/(1-\rho)}$ (defined by (11)). Then, by Lemma 3.5,

$$\int_I w^\varepsilon v' = \langle 2\varepsilon f^\varepsilon, v\rangle \ \ \forall v \in H_0^1(I).$$

Passing to the limit on ε gives, using (19) and Lemma 3.7,

$$\int_I wv' = \int_I \rho f v \quad \forall v \in H_0^1(I).$$

Finally, for any $v \in H_{1/(1-\rho)}$,

$$\begin{aligned}\int_I \rho f v = \int_I wv' &= \int_{\rho\neq 1} wv' \\ &= \int_{\rho\neq 1} \frac{(1-\rho)w}{1-\rho}v' = \int_{\rho\neq 1}\frac{\partial_x u}{1-\rho}v'\end{aligned}$$

and we conclude that u is the solution to (13).

So, we proved that there exists a subsequence of $(u^\varepsilon)_\varepsilon$ converging to u as ε tends to zero. Since the same work can be done for each subsequence of $(u^\varepsilon)_\varepsilon$, we conclude that $(u^\varepsilon)_\varepsilon$ itself converges to u, which completes the proof of the theorem. $\square$

Proof of Lemma 3.5: $-\partial_x (w^\varepsilon) = 2\varepsilon f^\varepsilon$.

First, a simple computation shows that

$$\partial_x w^\varepsilon = \sum_{\substack{i\,\in\,1\ldots N^\varepsilon-1\\ d_i\,>\,0,\ d_{i+1}\,>\,0}} \frac{1}{2\varepsilon}\left(\frac{u_{i+1}^\varepsilon - u_i^\varepsilon}{d_{i+1}^\varepsilon} - \frac{u_i^\varepsilon - u_{i-1}^\varepsilon}{d_i^\varepsilon}\right)\mathbf{1}_{]q_i^\varepsilon-\varepsilon,q_i^\varepsilon+\varepsilon[}$$

$$+\sum_{k=1}^{P}\sum_{i=i_k+1}^{i_k+N_k-1}\frac{1}{2\varepsilon}\left(\beta_{i+1}^\varepsilon - \beta_i^\varepsilon\right)\mathbf{1}_{]q_i^\varepsilon-\varepsilon,q_i^\varepsilon+\varepsilon[}$$

$$+\sum_{k=1}^{P}\frac{1}{2\varepsilon}\left(\beta_{i_k+1}^\varepsilon - \frac{u_{i_k}^\varepsilon - u_{i_{k-1}}^\varepsilon}{d_{i_k}^\varepsilon}\right)\mathbf{1}_{]q_{i_k}^\varepsilon-\varepsilon,q_{i_k}^\varepsilon+\varepsilon[}$$

$$+\sum_{k=1}^{P}\frac{1}{2\varepsilon}\left(\frac{u_{i_k+N_k+1}^\varepsilon - u_{i_k+N_k}^\varepsilon}{d_{i_k+N_k+1}^\varepsilon} - \beta_{i_k+N_k}^\varepsilon\right)\mathbf{1}_{]q_{i_k+N_k}^\varepsilon-\varepsilon,q_{i_k+N_k}^\varepsilon+\varepsilon[}.$$

Then, combining this with system (7-8) and with the definition of β_i (17), we get

$$\partial_x w^\varepsilon = -\sum_{i=1}^{N^\varepsilon-1} f_i^\varepsilon \mathbf{1}_{]q_i^\varepsilon-\varepsilon,q_i^\varepsilon+\varepsilon[}.$$

$\square$

Proof of Lemma 3.6: $\rho^\varepsilon \xrightarrow{\star} \rho$ in $L^\infty(I)$.

Since $\rho^\varepsilon - \rho = (\rho^\varepsilon - \chi^\varepsilon) + (\chi^\varepsilon - \rho)$ and $\chi^\varepsilon \xrightarrow{\star} \rho$ in $L^\infty(I)$, the result will follow provided we show that

$$\forall \varphi \in L^1(I)\ ,\ \lim_{\varepsilon\to 0}\left(\int_I \chi^\varepsilon\varphi - \int_I \rho^\varepsilon\varphi\right) = 0.$$

By density of the set of stairs functions in $L^1(I)$, and by using the fact that $(\rho^\varepsilon)_\varepsilon$ and $(\chi^\varepsilon)_\varepsilon$ are bounded in $L^\infty(I)$, this in turn will follow from

$$\forall \varphi \text{ piecewise constant on } I\ ,\ \lim_{\varepsilon\to 0}\left(\int_I \chi^\varepsilon\varphi - \int_I \rho^\varepsilon\varphi\right) = 0.$$

To show this, it suffices to prove that

$$\forall \alpha,\beta\ ,\ \ 0<\alpha<\beta<1\ ,\ \lim_{\varepsilon\to 0}\left(\int_\alpha^\beta \chi^\varepsilon - \int_\alpha^\beta \rho^\varepsilon\right) = 0.$$

In order to do so take α and β such that $0<\alpha<\beta<1$ and denote the particules whose centers are in $[\alpha,\beta]$ by $q_{i_0}^\varepsilon$, $q_{i_0+1}^\varepsilon$, $\ldots$, $q_{j_0}^\varepsilon$ (again we drop the ε superscript, keeping in mind that i_0 and j_0 depend on ε). Since $\int_{q_i^\varepsilon}^{q_{i+1}^\varepsilon}(\chi^\varepsilon - \rho^\varepsilon) = 2\varepsilon - (q_{i+1}^\varepsilon - q_i^\varepsilon - d_{i+1}^\varepsilon) = 0$ for $1 \le i \le N^\varepsilon - 1$, we have

$$\int_\alpha^\beta \chi^\varepsilon - \int_\alpha^\beta \rho^\varepsilon = \int_\alpha^{q_{i_0}^\varepsilon}(\chi^\varepsilon - \rho^\varepsilon) - \int_{q_{j_0}^\varepsilon}^\beta(\chi^\varepsilon - \rho^\varepsilon).$$

Then, a simple computation shows that the left-hand side converges to zero as ε tends to zero wich completes the proof of the lemma. $\square$

Proof of Lemma 3.7: $f^\varepsilon \overset{\star}{\rightharpoonup} \rho f$ in $L^\infty(I)$.

Writing

$$f^\varepsilon - \rho f = (f^\varepsilon - \chi^\varepsilon f) + (\chi^\varepsilon f - \rho f)\,.$$

and using the fact that $\chi^\varepsilon \overset{\star}{\rightharpoonup} \rho$ in $L^\infty(I)$, the required result will follow as soon as we prove that

$$\forall \varphi \in L^1(I)\ ,\ \lim_{\varepsilon\to 0} \int_I (f^\varepsilon - \chi^\varepsilon f)\,\varphi = 0,$$

To obtain this, merely compute

$$\begin{aligned}\int_I (f^\varepsilon - \chi^\varepsilon f)\,\varphi \quad &= \quad \sum_{i=1}^{N^\varepsilon-1} \int_{q_i^\varepsilon-\varepsilon}^{q_i^\varepsilon+\varepsilon} [f_i^\varepsilon - f(x)]\varphi(x)dx \\ & \quad - \int_0^\varepsilon f(x)\varphi(x)dx - \int_{1-\varepsilon}^1 f(x)\varphi(x)dx\end{aligned}$$

and, since the last two terms tend to zero as ε go to zero, the result will follow as soon as we prove

$$\lim_{\varepsilon\to 0} \sum_{i=1}^{N^\varepsilon-1} \int_{q_i^\varepsilon-\varepsilon}^{q_i^\varepsilon+\varepsilon} [f_i^\varepsilon - f(x)]\varphi(x)dx = 0.$$

To do so, use the Lipschitz hypothesis on f to write

$$|f_i^\varepsilon - f(x)| = \frac{1}{2\varepsilon}\left| \int_{q_i^\varepsilon(t)-\varepsilon}^{q_i^\varepsilon(t)+\varepsilon} (f(y,t) - f(x,t))dy \right| \le 2C\varepsilon,$$

which gives

$$\left| \int_0^1 (f^\varepsilon - \chi^\varepsilon f)\varphi \right| \le 2C\varepsilon \|\varphi\|_{L^1(I)}$$

and completes the proof of the lemma. □

Proof of Lemma 3.8: $(u^\varepsilon)_\varepsilon$ is bounded in $H_0^1(I)$.

Using the fact that $\partial_x u^\varepsilon$ is zero when ρ^ε is equal to 1 we obtain

$$\|u^\varepsilon\|^2_{H_0^1(I)} = \int_{I\cap\{\rho^\varepsilon<1\}} |\partial_x u^\varepsilon|^2 \le \int_{I\cap\{\rho^\varepsilon<1\}} \frac{1}{1-\rho^\varepsilon} |\partial_x u^\varepsilon|^2$$

and

$$\|u^\varepsilon\|^2_{H_0^1(I)} \le \int_{I\cap\{\rho^\varepsilon<1\}} \tilde{w}^\varepsilon \partial_x u^\varepsilon = \int_I \tilde{w}^\varepsilon \partial_x u^\varepsilon, \tag{20}$$

where $\tilde{w}^\varepsilon$ is the function, piecewise constant, with $\tilde{w}^\varepsilon = w_i^\varepsilon$ on $[q_{i-1}^\varepsilon(t), q_i^\varepsilon(t)]$.

A simple computation shows that, in the sense of distribution, we have

$$\partial_x \tilde{w}^\varepsilon = \sum_{\substack{i\in 1..N^\varepsilon-1 \\ d_i^\varepsilon>0,\ d_{i+1}^\varepsilon>0}} \left(\frac{u_{i+1}^\varepsilon - u_i^\varepsilon}{d_{i+1}^\varepsilon} - \frac{u_i^\varepsilon - u_{i-1}^\varepsilon}{d_i^\varepsilon} \right) \delta_{q_i^\varepsilon} + \sum_{k=1}^{N_c} \sum_{i=i_k+1}^{i_k+N_k-1} \left(\beta_{i+1}^\varepsilon - \beta_i^\varepsilon\right) \delta_{q_i^\varepsilon}$$

$$+\sum_{k=1}^{N_c}\left(\beta^\varepsilon_{i_k+1}-\frac{u^\varepsilon_{i_k}-u^\varepsilon_{i_{k-1}}}{d_{i_k}}\right)\delta_{q^\varepsilon_{i_k}}+\sum_{k=1}^{N_c}\left(\frac{u^\varepsilon_{i_k+N_k+1}-u^\varepsilon_{i_k+N_k}}{d_{i_k+N_k+1}}-\beta^\varepsilon_{i_k+N_k}\right)\delta_{q^\varepsilon_{i_k+N_k}}$$

Then, combining this with system (7-8) and with the definition of β_i (17), we get

$$\partial_x\tilde{w}^\varepsilon=-\sum_{i=1}^{N^\varepsilon-1}2\varepsilon f_i^\varepsilon\delta_{q_i^\varepsilon},$$

which can be extended to $H^{-1}(I)$ by density. This, with (20) and the fact that $\mathbf{q}^\varepsilon$ is ε-feasible, gives

$$\|u^\varepsilon\|^2_{H^1_0(I)}\leq\sum_{i=1}^{N^\varepsilon-1}2\varepsilon f_i^\varepsilon u^\varepsilon(q_i^\varepsilon)\leq\|u^\varepsilon\|_{L^\infty(I)}\|f\|_{L^1(I)}.$$

and, by continuous injection of $H^1_0(I)$ in $\mathcal{C}^0(I)$ we obtain

$$\|u^\varepsilon\|_{H^1_0(I)}\leq C\|f\|_{L^1(I)}$$

which completes the proof of the lemma. □

Proof of Lemma 3.9: $(w^\varepsilon)_\varepsilon$ and $(\partial_x w^\varepsilon)_\varepsilon$ are bounded in $L^\infty(I)$.

By (15-16), it suffices to obtain an upper-bound for $(w_i^\varepsilon)_i$, where

$$w_i^\varepsilon=\frac{1}{1-\rho_i^\varepsilon}\partial_x u^\varepsilon=\frac{u_i^\varepsilon-u_{i-1}^\varepsilon}{d_i^\varepsilon}\text{ if }\rho_i^\varepsilon<1,$$

and $w_i^\varepsilon=\beta_i^\varepsilon$ otherwise. Note that, by (17) and the fact that $\mathbf{q}^\varepsilon$ is ε-feasible

$$\forall k=1\ldots N_c\ ,\ \ \forall j=1\ldots N_k\ ,\ \ \left|\beta^\varepsilon_{i_k+j}\right|=\left|w^\varepsilon_{i_k}-2\varepsilon\sum_{m=i_k}^{i_k+j-1}f_m\right|\leq\left|w^\varepsilon_{i_k}\right|+\|f\|_{L^1(I)}.$$

Therefore, to show that $\|w^\varepsilon\|_{L^\infty(I)}$ is bounded it suffices to prove that $(w_i^\varepsilon)_{\{i\text{ s.t. }\rho_i<1\}}$ is bounded. In order to do so, a simple computation gives using (6)

$$\forall i\text{ s.t. }\rho_i<1\ ,\ \ w_i^\varepsilon=2\varepsilon\sum_{k=i+1}^{N^\varepsilon-1}\frac{D_{N^\varepsilon}-D_k}{D_{N^\varepsilon}}f_k^\varepsilon-2\varepsilon\sum_{k=1}^{i}\frac{D_k}{D_{N^\varepsilon}}f_k^\varepsilon. \tag{21}$$

Then, from $\frac{D_{N^\varepsilon}-D_k}{D_{N^\varepsilon}}\leq 1$, $\frac{D_k}{D_{N^\varepsilon}}\leq 1$ and the fact that $\mathbf{q}^\varepsilon$ is ε-feasible it follows that

$$|w_i^\varepsilon|\leq\sum_{k=1}^{N^\varepsilon-1}|2\varepsilon f_k^\varepsilon|\leq\|f\|_{L^1(I)}$$

and we conclude that

$$\|w^\varepsilon\|_{L^\infty(I)}\leq 2\|f\|_{L^1(I)}.$$

The bound on $(\partial_x w^\varepsilon)_\varepsilon$ is easily obtained using lemma 3.5 combined with $f\in L^\infty(I)$, which completes the proof of the lemma. □

Proof of Lemma 3.10: $\partial_x u^\varepsilon \overset{\star}{\rightharpoonup} (1-\rho)w$ in $L^\infty(I)$.

Writing

$$\partial_x u^\varepsilon - (1-\rho)w = \{\partial_x u^\varepsilon - (1-\rho^\varepsilon)w^\varepsilon\} + \{(1-\rho^\varepsilon)(w^\varepsilon - w)\} + \{((1-\rho^\varepsilon)-(1-\rho))\, w\}\,,$$

we shall prove the weak-star convergence to zero in $L^\infty(I)$ of each term of the right-hand side.

The first term goes strongly to zero in $L^\infty(I)$. Indeed, using the definitions of w^ε (15), ρ^ε (14) and u^ε (9) we obtain

$$\|\partial_x u^\varepsilon - (1-\rho^\varepsilon)w^\varepsilon\|_{L^\infty(I)} \le \frac{1}{2}\sup_i |w^\varepsilon_{i+1} - w^\varepsilon_i|.$$

This, combined with $|w^\varepsilon_{i+1} - w^\varepsilon_i| = |2\varepsilon f^\varepsilon_i|$ gives

$$\|\partial_x u^\varepsilon - (1-\rho^\varepsilon)w^\varepsilon\|_{L^\infty(I)} \le \varepsilon\|f\|_{L^\infty(I)},$$

which proves the uniform convergence to zero.

To prove the weak-star convergence of the second term, by density of $\mathcal{C}^\infty_0(I)$ in $L^1(I)$ and lemma 3.9, it suffices to take test-functions φ in $\mathcal{C}^\infty_0(I)$. Therefore, the result follows immediately from

$$\left|\int_I (1-\rho^\varepsilon)(w^\varepsilon - w)\varphi\right| \le |I|\|\varphi\|_{L^\infty(I)}\|w^\varepsilon - w\|_{L^\infty(I)}$$

together with (19).

We shall now prove the weak-star convergence of the last term to zero. To do so, take φ in $L^1(I)$. Using that $w \in \mathcal{C}^0(I)$, it follows that $w\varphi \in L^1(I)$ and by Lemma 3.6

$$\lim_{\varepsilon\to 0}\int_I ((1-\rho^\varepsilon)-(1-\rho))\, w\varphi = 0,$$

as required. □

4. Non-stationary model

In this section, we consider the non-stationnary lubrication model without inertia. We suppose that the extremal particles are fixed ($q_0(t) = 0$, $q_N(t) = 1$, $\forall t$). In that case, the unknowns are the positions of the particles 1 to $N-1$ against time. We denote by $\mathbf{q}(t) = (q_1(t), \ldots, q_{N-1}(t))$ these positions. Since the balance of forces is achieved at each instant t, $\mathbf{q}$ is solution to

$$\left|\begin{array}{l} -A(\mathbf{q})\dot{\mathbf{q}} + \mathbf{f}(t,\mathbf{q}) = 0, \\ \mathbf{q}(0) = \mathbf{q}_0, \end{array}\right. \tag{22}$$

where A is defined in (4) and $\mathbf{f} = (f_i)_i$ is the vector made of the external forces exerted on the particles.

Proposition 4.1. *Suppose that* $f_i \in L^1_{loc}(\mathbb{R}_+, L^\infty(\mathbb{R}^N))$ *and is Lipschitz with respect to the second variable. Let* $\mathbf{q}_0$ *be a strictly* ε*-feasible vector (see definition 2.1). Then, the system of ODE (22) has a unique global solution.*

Proof. Equation (22) can be written $\dot{\mathbf{q}} = F(t, \mathbf{q})$ with $F(t, \mathbf{q}) = A^{-1}(\mathbf{q})\mathbf{f}(t, \mathbf{q})$. We apply Cauchy-Lipschitz theorem to this first order ODE on the set of strictly ε-feasible configurations

$$\Omega_\varepsilon = \{\mathbf{q}\ ,\ \ d_i = q_i - q_{i-1} - 2\varepsilon > 0\ \forall i \in \{1 \dots N\}\}\,.$$

If the maximal solution were defined over $[0, T^*[$ with $T^* < +\infty$, we could find an index i and a subsequence $(t_n)_n$ such that

$$d_i(t_n) \to 0 \text{ when } n \to +\infty \text{ and } t_n \to T^*.$$

The extremal spheres being fixed, there also exists k such that $d_k(t_n) \geq \eta > 0$ for all n. By summing up lines i to $k-1$ of equation (22) and integrating it over $[0, t_n]$ we obtain

$$C + \sum_{j=i}^{k-1} (\ln(d_k(t_n)) - \ln(d_i(t_n))) = -\sum_{j=i}^{k-1} \int_0^{t_n} f_j(s, \mathbf{q}(s))ds,$$

where C is a constant. This is a contradiction since the left-hand side goes to infinity when $n \to +\infty$ while the right-hand side is bounded. Consequently, $T^* = +\infty$ and the solution is global. $\square$

Similarly to what has been done in the previous section, we can extend this approach to ε-feasible situations in the large sense. We denote by $\mathbf{u}$ the velocity $\dot{\mathbf{q}}$ of the particles.

Proposition 4.2. *Suppose that* $f_i \in L^1_{loc}(\mathbb{R}_+, L^\infty(\mathbb{R}^N))$ *and is Lipschitz with respect to the second variable. Let* $\mathbf{q}_0$ *be an* ε*-feasible vector. The system of ODE (7-8) with initial condition* $\mathbf{q}(0) = \mathbf{q}_0$ *has a unique global solution* $t \to \mathbf{q}(t)$*. This solution will be denoted by* $\mathbf{q} = \mathcal{Q}(\mathbf{q}_0, \mathbf{f}, \varepsilon)$*. The velocity* $\mathbf{u} = \dot{\mathbf{q}}$ *can be written with respect to* $\mathbf{q}$ *using (6).*

The continuous counterpart of this non-stationnary model is the following system of PDEs made of the constitutive law derived in the previous section and the transport equation:

$$\left|\begin{array}{l} -\partial_x \left(\dfrac{1}{1-\rho}\partial_x u\right) = \rho f \\ \partial_t \rho + \partial_x(\rho u) = 0 \end{array}\right.$$

To show the convergence of the discrete model (22) to this continuous model, we construct a micro-to-macro operator $\tilde{\mathcal{Q}}$. It is obtained from $\mathcal{Q}$ in a similar way than $\tilde{\mathcal{P}}$ is defined from $\mathcal{P}$ (see (9)). Given $T > 0$, an ε-feasible initial position vector $\mathbf{q}_0 \in \mathbb{R}^{N-1}$ and a force density $f \in L^1(]0, T[\times I)$, we define $\mathbf{q}$ as $\mathcal{Q}(\mathbf{q}_0, \mathbf{f}^\varepsilon, \varepsilon)$, where $\mathbf{f}^\varepsilon$ is given by

$$f_i^\varepsilon(t) = \frac{1}{2\varepsilon}\int_{q_i(t)-\varepsilon}^{q_i(t)+\varepsilon} f(t, s)ds.$$

We denote by $\mathbf{u} = \dot{\mathbf{q}}$ the velocities of the particles. Similarly to what has been done in the previous section (see (9) and (10)), for each time t we associate to vectors $\mathbf{q}(t)$ and

$\mathbf{u}(t)$ the functions $\chi(t,\cdot)$ and $u(t,\cdot)$ respectively constant by part and affine by part (see figures 7 and 8). We denote this mapping by $\tilde{\mathcal{Q}}$: $(\chi, u) = \tilde{\mathcal{Q}}(\bar{\mathbf{q}}, f, \varepsilon)$.

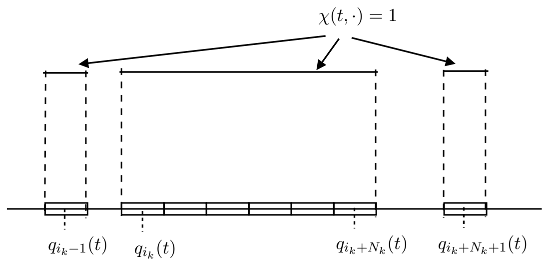

FIGURE 7. Solid phase function at time t: $t \longrightarrow \chi(t,.)$.

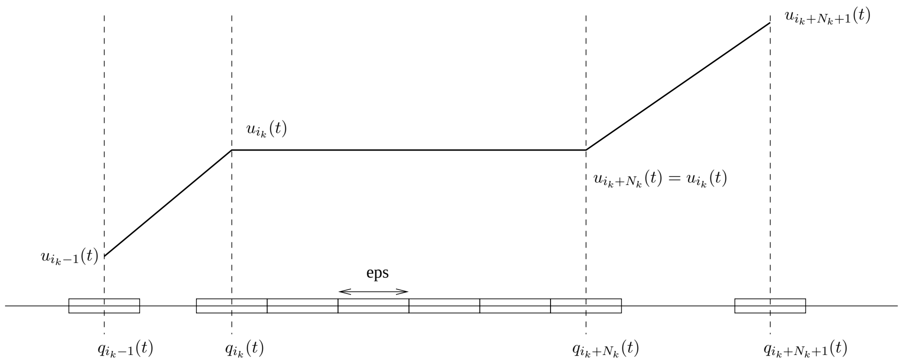

FIGURE 8. Velocity function at time t: $t \longrightarrow u(t,.)$.

To state the convergence result it remains to say what is meant by being a solution to the transport equation:

Definition 4.3. If $\rho_0 \in L^\infty(I)$, we say that

$$(\rho, u) \in L^\infty(]0,T[\times I) \times L^1(]0,T[\times I)$$

is a weak solution to problem

$$\left| \begin{array}{l} \partial_t \rho + \partial_x(\rho u) = 0, \\ \rho(0,\cdot) = \rho_0, \end{array} \right.$$

if, for all $\phi \in \mathcal{D}([0,T[\times I)$,

$$\int_0^T \int_I \rho(t,x)\partial_t\phi(t,x)dxdt + \int_I \rho_0(x)\phi(0,x)dx + \int_0^T \int_I \rho(t,x)u(t,x)\partial_x\phi(t,x)dxdt = 0.$$

The convergence result in the non-stationary case is the following:

Theorem 4.4. *Let $T > 0$ be given and f and ρ_0 be two measurable functions on $]0,T[\times I$ and I respectively. We suppose:*

$$f \in L^\infty(]0,T[, W^{1,\infty}(I)) \cap W^{1,\infty}(]0,T[, L^1(I)). \tag{23}$$

$$\rho_0 \in L^\infty(I) \text{ and } 0 \le \rho_0(x) \le 1 \quad \text{a.e. in } I. \tag{24}$$

Let $(\mathbf{q}_0^\varepsilon)_\varepsilon$ be a sequence of ε-feasible vectors (see definition 2.1), such that

$$\chi(\mathbf{q}_0^\varepsilon, \varepsilon) \overset{\star}{\rightharpoonup} \rho_0 \text{ in } L^\infty(I) \tag{25}$$

where $\chi(\mathbf{q}_0^\varepsilon, \varepsilon)$ is defined by (10). We denote by $(\chi^\varepsilon, u^\varepsilon) = \tilde{\mathcal{Q}}(\mathbf{q}_0^\varepsilon, f, \varepsilon)$ the solution to the discrete problem.

There exists $\rho \in L^\infty(]0,T[\times I)$ and $u \in L^\infty(]0,T[, H_0^1(I)) \cap W^{1,\infty}(]0,T[, L^\infty(I))$, such that, up to a subsequence,

$$\begin{aligned} &\chi^\varepsilon \overset{\star}{\rightharpoonup} \rho \text{ in } L^\infty(]0,T[\times I), \\ &u^\varepsilon \overset{\star}{\rightharpoonup} u \text{ in } L^\infty(]0,T[, H_0^1(I)), \\ &\partial_t u^\varepsilon \overset{\star}{\rightharpoonup} \partial_t u \text{ in } L^\infty(]0,T[\times I). \end{aligned}$$

Moreover, ρ and u verify

$$0 \le \rho(t,x) \le 1 \text{ for a.e. } (x,t) \in]0,T[\times I \tag{26}$$

$$-\partial_x\left(\frac{1}{1-\rho(t,\cdot)}\partial_x u(t,\cdot)\right) = \rho(t,\cdot) f(t,\cdot) \text{ in the sense of Prop. 3.1, for a.e. } t \in]0,T[\tag{27}$$

$$\partial_t \rho + \partial_x(\rho u) = 0\ , \quad \rho(0,\cdot) = \rho_0 \text{ in the sense of Def. 4.3.} \tag{28}$$

Proof. The proof of this theorem is similar to the one of theorem 3.2. The main difference is that now, we need to control the time regularity of the functions involved in order to pass to the limit when ε goes to zero, and to check that the transport equation is verified. The sketch of the proof is therefore similar to the one of theorem 3.2. Some of the lemmas involved here are even direct consequences of the computations made to prove their stationnary counterpart. These lemmas will be noticed in the following. For readability reasons, the proofs of the remaining lemmas will be postponed to the end of the section.

To begin, note that hypothesis (23) for f, together with proposition 4.2, gives the existence of a unique global solution to the discrete problem.

The fact that $|\chi^\varepsilon|$ is bounded by 1 immediatly gives

Lemma 4.5.

$$\exists \rho \in L^\infty(]0,T[\times I) \text{ such that } \chi^\varepsilon \overset{\star}{\rightharpoonup} \rho \text{ in } L^\infty(]0,T[\times I)$$

Moreover, since $0 \le \chi^\varepsilon \le 1$, we obtain (26).

Similarly to what has been done for the stationary case (see (14) and (15)), we define for each time t, $\rho^\varepsilon(t,\cdot)$ and $w^\varepsilon(t,\cdot)$ (see figures 9 and 10)

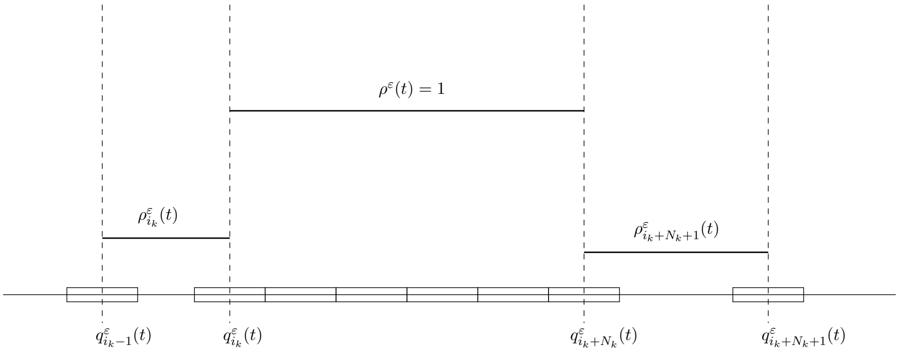

FIGURE 9. Density function: $t \to \rho^\varepsilon(t, .)$.

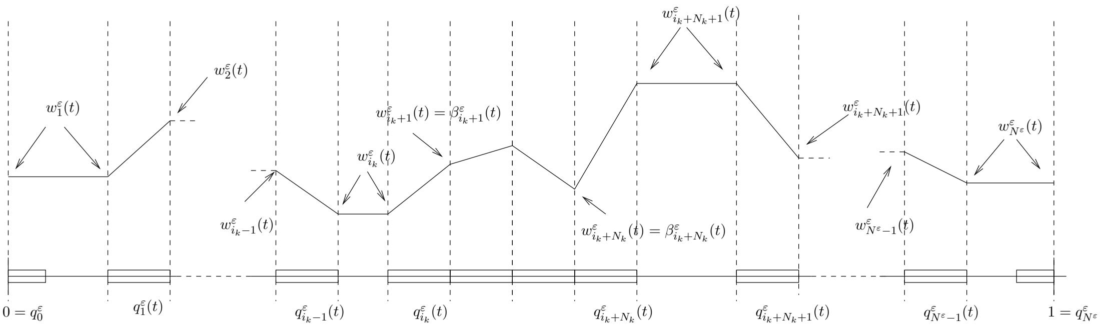

FIGURE 10. $t \to w^\varepsilon(t, .)$.

From the computations carried out in the proof of lemma 3.5 we get that (27) is true at the discrete level:

Lemma 4.6. *For any $t \in]0, T[$ and $\varepsilon > 0$,*

$$-\partial_x w^\varepsilon(t, \cdot) = f^\varepsilon(t, \cdot), \tag{29}$$

where $f^\varepsilon(t, \cdot) = \displaystyle\sum_{i=1}^{N^\varepsilon - 1} f_i^\varepsilon(t) \mathbf{1}_{]q_i^\varepsilon(t) - \varepsilon, q_i^\varepsilon(t) + \varepsilon[}$.

The following lemma shows that the transport equation (28) is also true at the discrete level:

Lemma 4.7. *For any $\varepsilon > 0$,*

$$\partial_t \rho^\varepsilon + \partial_x (\rho^\varepsilon u^\varepsilon) = 0, \tag{30}$$

in the sense of definition 4.3, with initial condition $\rho^\varepsilon(0, \cdot)$.

Remark 4.8. This result follows from the definition of ρ^ε and u^ε with respect to $\mathbf{q}^\varepsilon$.

The proofs of lemmas 3.6, 3.7 and 3.8 extend straightforwardly to the non-stationary case, which yields

$$\rho^\varepsilon \overset{\star}{\rightharpoonup} \rho \text{ in } L^\infty(]0, T[\times I) \text{ and } \rho^\varepsilon(0, .) \overset{\star}{\rightharpoonup} \rho_0 \text{ in } L^\infty(I) \tag{31}$$

$$f^\varepsilon \overset{\star}{\rightharpoonup} \rho f \text{in } L^\infty(]0,T[\times I),$$

(32) $$(u^\varepsilon)_\varepsilon \text{ is bounded in } L^\infty(]0,T[,H_0^1(I)).$$

From (32), we get the existence of a subsequence, still denoted by $(u^\varepsilon)_\varepsilon$ such that, when ε goes to zero,

(33) $$u^\varepsilon \overset{\star}{\rightharpoonup} u \text{ in } L^\infty(]0,T[,H_0^1(I)),$$

(34) $$\partial_x u^\varepsilon \overset{\star}{\rightharpoonup} \partial_x u \text{ dans } L^\infty(]0,T[,L^2(I)).$$

Similarly to the stationary case, in order to pass to the limit in the left-hand side of (27), we have to study the convergence of $(w^\varepsilon)_\varepsilon$. The same computations as in the proof of lemma 3.9 give:

Lemma 4.9. *$(w^\varepsilon)_\varepsilon$ and $(\partial_x w^\varepsilon)_\varepsilon$ are bounded in $L^\infty(]0,T[\times I)$.*

To obtain a strong convergence for w^ε, we need to control its time derivative, which will follow from the next lemma.

Lemma 4.10. *$(\partial_t w^\varepsilon)_\varepsilon$ is bounded $L^\infty(]0,T[\times I)$.*

From lemmas 4.9 and 4.10, together with Ascoli theorem, we get the existence of $w \in C^0(]0,T[\times I)$ and a subsequence still denoted $(w^\varepsilon)_\varepsilon$ such that

(35) $$w^\varepsilon \longrightarrow w \text{ in } \mathcal{C}^0(]0,T[\times I) \text{ when } \varepsilon \to 0.$$

From this result of strong convergence we can establish, as in the previous section, the following lemma:

Lemma 4.11. *$\partial_x u^\varepsilon \overset{\star}{\rightharpoonup} (1-\rho)w$ in $L^\infty(]0,T[\times I)$.*

Finally, similarily to the stationary case, we pass to the limit in (29) to obtain (27) by using test functions of type $(x,t) \to v(x)\Phi(t)$ with $v \in H_0^1(I)$ and $\Phi \in \mathcal{D}(]0,T[)$.

To finish the proof of the theorem, it remains to check (28) by passing to the limit in (30). To do so a strong convergence of $(u^\varepsilon)_\varepsilon$ is required, which will follow from the next lemma:

Lemma 4.12. *$(\partial_t u^\varepsilon)_\varepsilon$ is bounded in $L^\infty(]0,T[\times I)$.*

Using this last lemma, together with the fact that $(u^\varepsilon)_\varepsilon$ is bounded in $L^\infty(]0,T[,H_0^1(I))$, we deduce that $(u^\varepsilon)_\varepsilon$ is bounded in $H^1(]0,T[\times I)$ and consequently, there exists a subsequence, still denoted by $(u^\varepsilon)_\varepsilon$, such that

(36) $$u^\varepsilon \longrightarrow u \text{ in } L^2(]0,T[\times I) \text{ when } \varepsilon \to 0.$$

We are now going to let ε go to zero in (30). Let Ψ be in $\mathcal{D}([0,T[\times I)$, from lemma 4.7, we have

$$\int_0^T \int_I \rho^\varepsilon \partial_t \Psi dxdt + \int_I \rho^\varepsilon(0,x)\Psi(0,x)dx + \int_0^T \int_I \rho^\varepsilon u^\varepsilon \partial_x \Psi dxdt = 0.$$

The convergence of the first two terms comes from (31). To study the last term, merely write

$$\left|\int_0^T \int_I \rho^\varepsilon u^\varepsilon \partial_x \Psi dxdt - \int_0^T \int_I \rho u \partial_x \Psi dxdt\right| \\ \leq \left|\int_0^T \int_I \rho^\varepsilon (u^\varepsilon - u)\partial_x \Psi dxdt\right| + \left|\int_0^T \int_I (\rho^\varepsilon - \rho) u \partial_x \Psi dxdt\right|,$$

and use (36) together with (31) to show that it converges to zero, which completes the proof of the theorem. □

Proof of Lemma 4.7: $\partial_t \rho^\varepsilon + \partial_x (\rho^\varepsilon u^\varepsilon) = 0$.

Let Φ be given in $\mathcal{D}([0,T[\times I)$. We first compute the time-derivative term. By using the definition (14) of ρ^ε we obtain:

$$\int_0^T \int_a^b \rho^\varepsilon(t,x)\partial_t \phi(t,x)dxdt = \sum_{i=1}^{N^\varepsilon} \int_0^T \int_a^b \rho_i^\varepsilon(t)\partial_t\phi(t,x)\mathbf{1}_{[q_{i-1}^\varepsilon(t), q_i^\varepsilon(t)]}(x)dxdt$$

which gives

$$\int_0^T \int_a^b \rho^\varepsilon(t,x)\partial_t \phi(t,x)dxdt = -\sum_{i=1}^{N^\varepsilon} \int_0^T \int_a^b (\rho_i^\varepsilon)'(t)\phi(t,x)\mathbf{1}_{[q_{i-1}^\varepsilon(t), q_i^\varepsilon(t)]}(x)dxdt \\ -\sum_{i=1}^{N^\varepsilon} \int_a^b \rho_i^\varepsilon(0)\phi(0,x)\mathbf{1}_{[q_{i-1}^\varepsilon(0), q_i^\varepsilon(0)]}(x)dx \\ -\sum_{i=1}^{N^\varepsilon} \int_0^T \left(\rho_i^\varepsilon(t)u_i^\varepsilon(t)\phi(t,q_i^\varepsilon(t)) - \rho_i^\varepsilon(t)u_{i-1}^\varepsilon(t)\phi(t,q_{i-1}^\varepsilon(t))\right) dt.$$

By shifting the indexes in the last sum and computing $(\rho_i^\varepsilon)'$, we obtain

$$\int_0^T \int_a^b \rho^\varepsilon(t,x)\partial_t \phi(t,x)dxdt = \\ \sum_{i=1}^{N^\varepsilon} \int_0^T \int_a^b \frac{2\varepsilon(u_i^\varepsilon(t) - u_{i-1}^\varepsilon(t))}{(q_i^\varepsilon(t) - q_{i-1}^\varepsilon(t))^2}\phi(t,x)\mathbf{1}_{[q_{i-1}^\varepsilon(t), q_i^\varepsilon(t)]}(x)dxdt \\ -\int_a^b \rho^\varepsilon(0)\phi(0,x)dx + \sum_{i=1}^{N^\varepsilon} \int_0^T (\rho_{i+1}^\varepsilon(t) - \rho_i^\varepsilon(t))u_i^\varepsilon(t)\phi(t,q_i^\varepsilon(t))dt. \tag{37}$$

To compute the x-derivative term, an integration by part on each sub-interval $[q_{i-1}^\varepsilon(t), q_i^\varepsilon(t)]$ gives

$$\begin{aligned} &\int_0^T \int_a^b \rho^\varepsilon(t,x) u^\varepsilon(t,x) \partial_x \phi(t,x) dx dt \\ &= \sum_{i=1}^{N^\varepsilon} \int_0^T (\rho_i^\varepsilon(t) - \rho_{i+1}^\varepsilon(t)) u_i^\varepsilon(t) \phi(t, q_i^\varepsilon(t)) dt \\ &\quad - \sum_{i=1}^{N^\varepsilon} \int_0^T \int_a^b \frac{2\varepsilon(u_i^\varepsilon(t) - u_{i-1}^\varepsilon(t))}{(q_i^\varepsilon(t) - q_{i-1}^\varepsilon(t))^2} \phi(t,x) \mathbf{1}_{[q_{i-1}^\varepsilon(t), q_i^\varepsilon(t)]}(x) dx dt. \end{aligned} \tag{38}$$

The proof of the lemma is completed by summing up (37) and (38). □

Proof of Lemma 4.10: $(\partial_t w^\varepsilon)_\varepsilon$ is bounded in $L^\infty(]0,T[\times I)$.

Using the definition (15) of w^ε (we recall that w_i^ε now depends on t), a simple computation gives:

$$\partial_t w^\varepsilon(x,t) = w_{\text{der,aff}}^\varepsilon(x,t) - w_{\text{der,cst}}^\varepsilon(x,t),$$

where, for each time t, $w_{\text{der,cst}}^\varepsilon(t,\cdot)$ is the piecewise constant function such that

$$w_{\text{der,cst}}^\varepsilon(x,t) = w_{\text{cst},i}^\varepsilon(t) \stackrel{\text{def}}{=} \frac{w_{i+1}^\varepsilon(t) - w_i^\varepsilon(t)}{2\varepsilon} u_i^\varepsilon(t) \quad \text{on} \quad]q_i^\varepsilon(t) - \varepsilon, q_i^\varepsilon(t) + \varepsilon[,$$

and $w_{\text{der,aff}}^\varepsilon(t,\cdot)$ is the piecewise affine function such that

$$\begin{aligned} w_{\text{der,aff}}^\varepsilon(q_i^\varepsilon(t) - \varepsilon, t) &= (w_i^\varepsilon)'(t), \\ w_{\text{der,aff}}^\varepsilon(q_i^\varepsilon(t) + \varepsilon, t) &= (w_{i+1}^\varepsilon)'(t). \end{aligned}$$

Therefore, to show that $(\partial_t w^\varepsilon)_\varepsilon$ is bounded in $L^\infty(]0,T[\times I)$, it suffices to prove that the sets $\{w_{\text{cst},i}^\varepsilon(t)\}_i$, and $\{(w_i^\varepsilon)'(t)\}_i$ are bounded independently of ε and t.

By continuous injection of $H_0^1(I)$ in $\mathcal{C}(I)$ and the boundedness of $(u^\varepsilon)_\varepsilon$ in $L^\infty(]0,T[, H_0^1(I))$, we get that $(u^\varepsilon)_\varepsilon$ is bounded in $L^\infty(]0,T[\times I)$ (we denote this bound by M). This, combined with the fact that $|w_{i+1}^\varepsilon(t) - w_i^\varepsilon(t)| = 2\varepsilon|f_i^\varepsilon(t)|$ gives

$$w_{\text{cst},i}^\varepsilon(t) \le \|u^\varepsilon\|_\infty |f_i^\varepsilon(t)| \le M\|f\|_\infty,$$

which gives a bound for $\{w_{\text{cst},i}^\varepsilon(t)\}_i$.

To end the proof, it remains to show that $\{(w_i^\varepsilon)'(t)\}_i$ is bounded. As for the proof of lemma 3.9, it suffices to show that

$$\{(w_i^\varepsilon)'(t)\}_{i, \text{ such that } \rho_i^\varepsilon < 1}$$

is bounded. From expression (21) of w_i^ε, together with the bound on u^ε and hypothesis (23) on f we get

$$|(w_i^\varepsilon)'(t)| \le M\|f\|_{L^\infty(]0,T[,L^1(I))} + MC(b-a) + \|\partial_t f\|_{L^\infty(]0,T[,L^1(I))}, \tag{39}$$

where C is the Lipschitz constant of f, which completes the proof of the lemma. □

Proof of Lemma 4.12: $(\partial_t u^\varepsilon)_\varepsilon$ is bounded in $L^\infty(]0,T[\times I)$.

As for the previous lemma, we compute $\partial_t u^\varepsilon$:

$$\partial_t u^\varepsilon = u^\varepsilon_{\mathrm{der,aff},1} - u^\varepsilon_{\mathrm{der,aff},2} - u^\varepsilon_{\mathrm{der,cste}},$$

where, for each time t, $u^\varepsilon_{\mathrm{der,aff},1}(t,\cdot)$ is the piecewise affine function such that

$$u^\varepsilon_{\mathrm{der,aff},1}(t, q^\varepsilon_i(t)) = (u^\varepsilon_i)'(t),$$

$u^\varepsilon_{\mathrm{der,aff},2}(t,\cdot)$ is the piecewise affine function such that

$$u^\varepsilon_{\mathrm{der,aff},2}(t, q^\varepsilon_i(t)) = \left(\frac{u^\varepsilon_{i+1}(t) - u^\varepsilon_i(t)}{q^\varepsilon_{i+1}(t) - q^\varepsilon_i(t)}\right)^2,$$

and $u^\varepsilon_{\mathrm{der,cst}}(t,\cdot)$ is the piecewise constant function such that

$$u^\varepsilon_{\mathrm{der,cst}}(t,x) = \frac{u^\varepsilon_{i+1}(t) - u^\varepsilon_i(t)}{q^\varepsilon_{i+1}(t) - q^\varepsilon_i(t)} u^\varepsilon_i(t) \quad \text{on} \quad]q^\varepsilon_i(t), q^\varepsilon_{i+1}(t)[.$$

Since u^ε is bounded (see proof of lemma 4.10), the fact that $u^\varepsilon_{\mathrm{der,aff},2}$ and $u^\varepsilon_{\mathrm{der,cst}}$ are bounded comes from

$$\left|\frac{u^\varepsilon_{i+1}(t) - u^\varepsilon_i(t)}{q^\varepsilon_{i+1}(t) - q^\varepsilon_i(t)}\right| = (1 - \rho^\varepsilon_i(t))\,|w^\varepsilon_i(t)| \leq |w^\varepsilon_i(t)| \leq \|w^\varepsilon\|_{L^\infty(]0,T[\times I)}, \tag{40}$$

together with lemma 4.9.

It remains to prove that $u^\varepsilon_{\mathrm{der,aff},1}$ is bounded. Since, for each t we have $u^\varepsilon_{\mathrm{der,aff},1}(t,0) = 0$, it suffices to show that $\partial_x u^\varepsilon_{\mathrm{der,aff},1}$ is bounded. This in turn will come provided we prove

$$\frac{(u^\varepsilon_i)'(t) - (u^\varepsilon_{i-1})'(t)}{q^\varepsilon_i(t) - q^\varepsilon_{i-1}(t)}$$

is bounded independently of t, ε and i. To do so, we compute the time-derivative of both side of the following equality

$$\frac{u^\varepsilon_i(t) - u^\varepsilon_{i-1}(t)}{q^\varepsilon_i(t) - q^\varepsilon_{i-1}(t)} = (1 - \rho^\varepsilon_i(t)) w^\varepsilon_i(t)$$

to obtain

$$\left|\frac{(u^\varepsilon_i)'(t) - (u^\varepsilon_{i-1})'(t)}{q^\varepsilon_i(t) - q^\varepsilon_{i-1}(t)}\right| \leq |w^\varepsilon_i(t)|^2 + |(w^\varepsilon_i)'(t)| + \left(\frac{u^\varepsilon_i(t) - u^\varepsilon_{i-1}(t)}{q^\varepsilon_i(t) - q^\varepsilon_{i-1}(t)}\right)^2.$$

Finally, from lemma 4.9 and inequalities (39) and (40), this last quantity is bounded, which completes the proof of the lemma. □

5. Extensions

A first extension to the approach we presented here consists in integrating inertial effects for the particles. Equation (5), which expresses instantaneous force balance, is replaced by Newton's law

$$\frac{d\mathbf{u}}{dt} = -A(\mathbf{q})\mathbf{u} + \mathbf{f}$$

Having the number of masses go to infinity, it is natural to expect some quite of 1d pressureless Navier-Stokes equation:

$$\partial_t(\rho u) + \partial_t(\rho u^2) - \partial_x \left(\frac{1}{1-\rho} \partial_x u \right) = \rho f,$$

coupled with the transport equation

$$\partial_t \rho + \partial_x(\rho u) = 0.$$

Now another natural question is the following: considering that fluid viscosity goes to 0 (so that the effective viscosity in the previous model is $\varepsilon/(1-\rho)$, with $\varepsilon \to 0$), what kind of limit model can be expected ? This very question has been addressed in [16] for the case of a single particle against a wall. As ε goes to 0, viscous effects are likely to disappear on unsaturated zones (where $\rho < 1$). Besides, as lubrication forces (even for a vanishing viscosity) prevent contact in finite time, a model of the following type could be expected

$$\left| \begin{array}{rcr} \partial_t \rho + \partial_x(\rho u) & = & 0 \\ \partial_t(\rho u) + \partial_x(\rho u^2) + \partial_x p & = & f \\ \partial_t \gamma + \partial_x(\gamma u) & = & -p \\ \gamma \leq 0 \ , \ \ \rho \leq 1 \ , \ \ \gamma(1-\rho) & = & 0 \end{array} \right.$$

where p is an unknown pressure-like field which may take negative values (which distinguishes this gluey model from the standard pressureless gas situation), and γ is a field which keeps track of the constraint history experienced by a fluid particle (see again [16] for more details on the gluey particle model).

Extension of this approach to higher dimensions is delicate. Firstly, even with strong assumptions on the shape of particles (discs or spheres), the notion of maximal density is fuzzy. It may depend on the local arrangements of grains, and this structure at the microscopic level has to be described in some way at the macroscopic scale. Besides, for the same reason of local geometrical complexity, the effects of lubrication forces cannot be expected to be described by a simple scalar (equivalent viscosity), but by a modification of the stress tensor. An asymptotic analysis is proposed in [7, 8] for the static problem, with a precise estimation of both shear and extensional viscosities, but the problem of homogenization of the evolution problem is still widely open.

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